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CONVERGENCE THEOREMS FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. The main aim of this paper is to study the strong convergence of finite-step iteration scheme for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of CAT(0) spaces. The said iteration scheme includes modified Mann and Ishikawa iterations, the three-step iteration scheme of Xu and Noor and the scheme of Khan, Domlo and Fukhar-ud-din as special cases in Banach spaces. Our results extend and generalize many known results from the previous work given in the existing literature.

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1. Introduction and Preliminaries

Let T be a self map on a nonempty subset K of a metric space (X, d). Denote the set of fixed points of T by $F(T) = \{x \in K : T(x) = x\}$. We say that a mapping $T: K \to K$ is said to be:

(1) asymptotically nonexpansive [2] if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad (1 + u_n) d(x, y), \tag{1}$$

for all $x, y \in K$ and $n \geq 1$;

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$d(T^n x, p) \quad \leq \quad (1 + u_n) d(x, p), \tag{2}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(3) generalized asymptotically quasi-nonexpansive [3] if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{u_n\}$ and $\{s_n\}$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} s_n$ such that

$$d(T^n x, p) \leq (1 + u_n) d(x, p) + s_n, \tag{3}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(4) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L\, d(x, y), \tag{4}$$

for all $x, y \in K$ and $n \geq 1$;

(5) semi-compact if for any bounded sequence $\{x_n\}$ in K with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

If in definition (3), $s_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

Let $\{x_n\}$ be a sequence in a metric space (X, d) and let K be a nonempty subset of X. We say that the sequence $\{x_n\}$ is:

(6) of monotone type (A) with respect to K if for each $p \in K$, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and

$$d(x_{n+1}, p) \leq (1 + a_n) d(x_n, p) + b_n, \tag{5}$$

(7) of monotone type (B) with respect to K if for each $p \in K$, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and

$$d(x_{n+1}, K) \leq (1 + a_n) d(x_n, K) + b_n, \tag{6}$$

(see also [34]).

From the above definitions, it is clear that a sequence of monotone type (A) is a sequence of monotone type (B) but the converse does not hold in general.

The purpose of this paper is to extend Khan-Domlo-Fukhar-ud-din's [4] results to a special kind of metric space, namely, $CAT(0)$ space.

CAT(0)space. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [16]), $\mathbb{R}$-trees (see [26]), Euclidean buildings (see [17]), the complex Hilbert ball with a hyperbolic metric (see [23]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [16].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [27, 28]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [15], [19]-[22], [24], [29]-[33] and the references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see,e.g., [16]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and let $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [16]).

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \;\; \leq \;\; d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{7}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [13]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \;\; \leq \;\; \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{8}$$

The inequality (8) is the (CN) inequality of Bruhat and Tits [18]. The above inequality was extended in [21] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha) y) \;\; \leq \;\; & \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ & -\alpha(1-\alpha) d^2(x, y) \end{aligned} \tag{9}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [[16], p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \;\; \leq \;\; \alpha d(z, x) + (1-\alpha) d(z, y), \tag{10}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha) y : \alpha \in [0, 1]\}$.

In view of the above inequality, $CAT(0)$ spaces have Takahashi's [14] convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha) y$. A subset K of a $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

Various iteration processes have been studied for an asymptotically nonexpansive mapping T (and their generalizations asymptotically quasi-nonexpansive map etc.)

on a convex subset K of a normed space E. Schu [9] considered the following modified Mann iteration

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \end{cases} \tag{11}$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$.

Fukhar-ud-din and Khan [1] have studied the modified Ishikawa iteration:

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_{n(1)})x_n + \alpha_{n(1)} T^n y_n, \\ \quad\quad y_n = (1-\alpha_{n(2)})x_n + \alpha_{n(2)} T^n x_n, \ n \geq 1, \end{cases} \tag{12}$$

where $0 \leq \alpha_{n(1)}, \alpha_{n(2)} \leq 1$, such that $\{\alpha_{n(1)}\}$ is bounded away from 0 and 1 and $\{\alpha_{n(2)}\}$ is bounded away from 1.

Xu and Noor [11] introduced and studied a three-step iteration scheme. Khan et al. [4] have defined a general iteration scheme for a finite family of maps which extend the scheme of Khan and Takahashi [5] and the three-step iteration scheme of Xu and Noor [11] simultaneously, as follows:

Throughout this paper, we will use $I = \{1, 2, \ldots, r\}$, where $r \geq 1$. Suppose that $\alpha_{in} \in [0, 1]$, $n \geq 1$ and $i \in I$. Let $\{T_i : i \in I\}$ be a family of asymptotically quasi-nonexpansive self-maps of K. Let $x_1 \in K$. The scheme introduced in [4] is

$$\begin{cases} x_{n+1} = (1-\alpha_{rn})x_n + \alpha_{rn} T_r^n y_{(r-1)n}, \\ y_{(r-1)n} = (1-\alpha_{(r-1)n})x_n + \alpha_{(r-1)n} T_{r-1}^n y_{(r-2)n}, \\ \quad \vdots \\ y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n} T_2^n y_{1n}, \\ y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n} T_1^n x_n, \ n \geq 1. \end{cases} \tag{13}$$

Very recently, inspired by the scheme (13) and the work in [4], Xiao et al. [10] have introduced $(r+1)$-step iteration scheme with error term and studied its strong convergence under weaker boundary conditions.

The existence of fixed (common fixed) points of one (or two maps or family of maps) is not known in many situations. So the approximation of fixed points (common fixed) of one or more nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive maps by various iterations have been extensively studied in Banach spaces, convex metric spaces and $CAT(0)$ spaces (see [1], [3]-[11], [12], [15], [20]-[21], [24]-[25], [29], [32]-[33]).

We now translate the scheme (13) from the normed space setting to the more general setup of $CAT(0)$ space as follows:

$$x_1 \in K, \quad x_{n+1} = U_{n(r)}x_n, \; n \geq 1, \tag{14}$$

where

$$\begin{cases} U_{n(0)} = I, \text{ the identity map,} \\ U_{n(1)}x = (1-\alpha_{n(1)})x \oplus \alpha_{n(1)}T_1^n U_{n(0)}x, \\ \quad\vdots \\ U_{n(r-1)}x = (1-\alpha_{n(r-1)})x \oplus \alpha_{n(r-1)}T_{r-1}^n U_{n(r-2)}x, \\ U_{n(r)}x = (1-\alpha_{n(r)})x \oplus \alpha_{n(r)}T_r^n U_{n(r-1)}x, \; n \geq 1, \end{cases}$$

where $0 \leq \alpha_{n(i)} \leq 1$ for each $i \in I$.

In a $CAT(0)$ space, the scheme (14) provides analogues of:

(i) the scheme (11) if $r = 1$ and $T_1 = T$;

(ii) the scheme (12) if $r = 2$ and $T_1 = T_2 = T$ and

(iii) the Xu and Noor [11] iteration scheme if $r = 3$, $T_1 = T_2 = T_3 = T$.

In this paper, we establish strong convergence theorems for the iteration scheme (14) to converge to common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of $CAT(0)$ spaces. Our result extends as well as refines the corresponding results of [1], [3]-[11], [24] and many others.

We need the following useful lemmas to prove our convergence results.

Lemma 1. *(See [31]) Let X be a $CAT(0)$ space.*

(i) For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t\, d(x, y) \quad \text{and} \quad d(y, z) = (1-t)\, d(x, y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x, y \in X$ *and* $t \in [0,1]$, *we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 2. *(See [4]) Let* $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ *be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1+q_n)p_n + r_n,\ n \geq 0,\ \sum_{n=0}^{\infty} q_n < \infty,\ \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(i) $\lim_{n\to\infty} p_n$ *exists.*

(ii) In addition, if $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

2. Main Results

In this section, we prove strong convergence theorems of finite-step iteration scheme (14) for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of $CAT(0)$ spaces.

Theorem 3. *Let* X *be a complete CAT(0) space and let* K *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in I\}$ *be a finite family of generalized asymptotically quasi-nonexpansive self-maps on* K *with sequences* $\{u_{n(i)}\}, \{s_{n(i)}\} \subset [0,\infty)$ *for each* $i \in I$, *respectively, such that* $\sum_{n=1}^{\infty} u_{n(i)} < \infty$ *and* $\sum_{n=1}^{\infty} s_{n(i)} < \infty$. *Assume that* $F = \cap_{i=1}^{r} F(T_i)$ *is closed. Let* $\{x_n\}$ *be the general iteration scheme defined by (14). Then the sequence* $\{x_n\}$ *is of monotone type* (A) *and monotone type* (B) *with respect to* F. *Moreover,* $\{x_n\}$ *converges strongly to a point in* F *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x,F) = \inf_{p\in F}\{d(x,p)\}$.

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F$, from (3), (14) and Lemma 2.1(ii), we have

$$\begin{aligned} d(U_{n(1)}x_n, p) &= d((1-\alpha_{n(1)})x_n \oplus \alpha_{n(1)}T_1^n x_n, p) \\ &\leq (1-\alpha_{n(1)})d(x_n,p) + \alpha_{n(1)}d(T_1^n x_n, p) \\ &\leq (1-\alpha_{n(1)})d(x_n,p) + \alpha_{n(1)}[(1+u_{n(1)})d(x_n,p) + s_{n(1)}] \\ &\leq (1+u_{n(1)})d(x_n,p) + \alpha_{n(1)}s_{n(1)} \\ &= (1+u_{n(1)})d(x_n,p) + A_{n(1)}, \end{aligned} \tag{15}$$

where $A_{n(1)} = \alpha_{n(1)} s_{n(1)}$, since by assumption $\sum_{n=1}^{\infty} s_{n(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{n(1)} < \infty$.

Again from (3), (14), Lemma 2.1(ii) and using (15), we obtain

$$
\begin{aligned}
d(U_{n(2)}x_n, p) &= d((1-\alpha_{n(2)})x_n \oplus \alpha_{n(2)} T_2^n U_{n(1)} x_n, p) \\
&\leq (1-\alpha_{n(2)}) d(x_n, p) + \alpha_{n(2)} d(T_2^n U_{n(1)} x_n, p) \\
&\leq (1-\alpha_{n(2)}) d(x_n, p) + \alpha_{n(2)} [(1+u_{n(2)}) d(U_{n(1)} x_n, p) + s_{n(2)}] \\
&\leq (1-\alpha_{n(2)}) d(x_n, p) + \alpha_{n(2)} (1+u_{n(2)}) d(U_{n(1)} x_n, p) + \alpha_{n(2)} s_{n(2)} \\
&\leq (1-\alpha_{n(2)}) d(x_n, p) + \alpha_{n(2)} (1+u_{n(2)}) [(1+u_{n(1)}) d(x_n, p) + A_{n(1)}] \\
&\quad + \alpha_{n(2)} s_{n(2)} \\
&\leq (1+u_{n(1)})(1+u_{n(2)}) d(x_n, p) + \alpha_{n(2)} (1+u_{n(2)}) A_{n(1)} + \alpha_{n(2)} s_{n(2)} \\
&= (1+u_{n(1)}+u_{n(2)}+u_{n(1)}u_{n(2)}) d(x_n, p) + \alpha_{n(2)} (1+u_{n(2)}) A_{n(1)} \\
&\quad + \alpha_{n(2)} s_{n(2)} \\
&\leq (1+t_{n(2)}) d(x_n, p) + A_{n(2)}, \qquad (16)
\end{aligned}
$$

where $t_{n(2)} = u_{n(1)} + u_{n(2)} + u_{n(1)}u_{n(2)}$ and $A_{n(2)} = \alpha_{n(2)}(1+u_{n(2)})A_{n(1)} + \alpha_{n(2)} s_{n(2)}$, since by assumptions $\sum_{n=1}^{\infty} u_{n(1)} < \infty$, $\sum_{n=1}^{\infty} u_{n(2)} < \infty$, $\sum_{n=1}^{\infty} s_{n(2)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} t_{n(2)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(2)} < \infty$.

Further using (3), (14), Lemma 2.1(ii) and (17), we obtain

$$
\begin{aligned}
d(U_{n(3)}x_n, p) &= d((1-\alpha_{n(3)})x_n \oplus \alpha_{n(3)} T_3^n U_{n(2)} x_n, p) \\
&\leq (1-\alpha_{n(3)}) d(x_n, p) + \alpha_{n(3)} d(T_3^n U_{n(2)} x_n, p) \\
&\leq (1-\alpha_{n(3)}) d(x_n, p) + \alpha_{n(3)} [(1+u_{n(3)}) d(U_{n(2)} x_n, p) + s_{n(3)}] \\
&\leq (1-\alpha_{n(3)}) d(x_n, p) + \alpha_{n(3)} (1+u_{n(3)}) d(U_{n(2)} x_n, p) + \alpha_{n(3)} s_{n(3)} \\
&\leq (1-\alpha_{n(3)}) d(x_n, p) + \alpha_{n(3)} (1+u_{n(3)}) [(1+t_{n(2)}) d(x_n, p) + A_{n(2)}] \\
&\quad + \alpha_{n(3)} s_{n(3)} \\
&\leq (1+u_{n(3)})(1+t_{n(1)}) d(x_n, p) + \alpha_{n(3)} (1+u_{n(3)}) A_{n(2)} + \alpha_{n(3)} s_{n(3)} \\
&= (1+u_{n(3)}+t_{n(2)}+u_{n(3)}t_{n(2)}) d(x_n, p) + \alpha_{n(3)} (1+u_{n(3)}) A_{n(2)} \\
&\quad + \alpha_{n(3)} s_{n(3)} \\
&\leq (1+t_{n(3)}) d(x_n, p) + A_{n(3)}, \qquad (17)
\end{aligned}
$$

where $t_{n(3)} = u_{n(3)} + t_{n(2)} + u_{n(3)}t_{n(2)}$ and $A_{n(3)} = \alpha_{n(3)}(1+u_{n(3)})A_{n(2)} + \alpha_{n(3)} s_{n(3)}$, since by assumptions $\sum_{n=1}^{\infty} u_{n(3)} < \infty$, $\sum_{n=1}^{\infty} t_{n(2)} < \infty$, $\sum_{n=1}^{\infty} s_{n(3)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(2)} < \infty$, it follows that $\sum_{n=1}^{\infty} t_{n(3)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(3)} < \infty$. Continuing the above process, using (3), (14) and Lemma 2.1(ii), we get

$$\begin{aligned}
d(x_{n+1},p) &= d((1-\alpha_{n(r)})x_n \oplus \alpha_{n(r)} T_r^n U_{n(r-1)} x_n, p) \\
&\leq (1-\alpha_{n(r)}) d(x_n,p) + \alpha_{n(r)} d(T_r^n U_{n(r-1)} x_n, p) \\
&\leq (1-\alpha_{n(r)}) d(x_n,p) + \alpha_{n(r)} [(1+u_{n(r)}) d(U_{n(r-1)} x_n, p) + s_{n(r)}] \\
&\leq (1-\alpha_{n(r)}) d(x_n,p) + \alpha_{n(r)} (1+u_{n(r)}) d(U_{n(r-1)} x_n, p) + \alpha_{n(r)} s_{n(r)} \\
&\leq (1-\alpha_{n(r)}) d(x_n,p) + \alpha_{n(r)} (1+u_{n(r)}) [(1+t_{n(r-1)}) d(x_n,p) + A_{n(r-1)}] \\
&\quad + \alpha_{n(r)} s_{n(r)} \\
&\leq (1+u_{n(r)})(1+t_{n(r-1)}) d(x_n,p) + \alpha_{n(r)}(1+u_{n(r)}) A_{n(r-1)} + \alpha_{n(r)} s_{n(r)} \\
&= (1+u_{n(r)}+t_{n(r-1)}+u_{n(r)}t_{n(r-1)}) d(x_n,p) + \alpha_{n(r)}(1+u_{n(r)}) A_{n(r-1)} \\
&\quad + \alpha_{n(r)} s_{n(r)} \\
&\leq (1+t_{n(r)}) d(x_n,p) + A_{n(r)}, \qquad (18)
\end{aligned}$$

where $t_{n(r)} = (1+u_{n(r)}+t_{n(r-1)}+u_{n(r)}t_{n(r-1)})$ and $A_{n(r)} = \alpha_{n(r)}(1+u_{n(r)})A_{n(r-1)} + \alpha_{n(r)} s_{n(r)}$, since by assumptions $\sum_{n=1}^{\infty} u_{n(r)} < \infty$, $\sum_{n=1}^{\infty} t_{n(r-1)} < \infty$, $\sum_{n=1}^{\infty} s_{n(r)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(r-1)} < \infty$, it follows that $\sum_{n=1}^{\infty} t_{n(r)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(r)} < \infty$. Now, from (18), we get

$$d(x_{n+1},p) \leq (1+t_{n(r)}) d(x_n,p) + A_{n(r)}, \qquad (19)$$

$$d(x_{n+1},F) \leq (1+t_{n(r)}) d(x_n,F) + A_{n(r)}. \qquad (20)$$

These inequalities, respectively, prove that $\{x_n\}$ is a sequence of monotone type (A) and monotone type (B) with respect to F.

Now, we prove that $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.

If $x_n \to p \in F$, then $\lim_{n\to\infty} d(x_n,p) = 0$. Since $0 \leq d(x_n,F) \leq d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F) = 0$. Applying Lemma 2.1 in equation (19), we have that $\lim_{n\to\infty} d(x_n,F)$ exists. Further, by hypothesis $\liminf_{n\to\infty}$ $d(x_n,F) = 0$, we conclude that $\lim_{n\to\infty} d(x_n,F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence.

Since $1+x \leq e^x$ for $x \geq 0$, therefore from (18), we have

$$\begin{aligned}
d(x_{n+m},p) &\leq (1+t_{n+m-1(r)})d(x_{n+m-1},p)+A_{n+m-1(r)} \\
&\leq e^{t_{n+m-1(r)}}d(x_{n+m-1},p)+A_{n+m-1(r)} \\
&\leq e^{t_{n+m-1(r)}}[e^{t_{n+m-2(r)}}d(x_{n+m-2},p)+A_{n+m-2(r)}] \\
&\quad +A_{n+m-1(r)} \\
&\leq e^{\{t_{n+m-1(r)}+t_{n+m-2(r)}\}}d(x_{n+m-2},p) \\
&\quad +e^{t_{n+m-1(r)}}[A_{n+m-2(r)}+A_{n+m-1(r)}] \\
&\leq \dots \\
&\leq \Big\{e^{\sum_{k=n}^{n+m-1}t_{k(r)}}\Big\}d(x_n,p)+\Big\{e^{\sum_{k=n+1}^{n+m-1}t_{k(r)}}\Big\}\Big(\sum_{k=n}^{n+m-1}A_{k(r)}\Big) \\
&\leq \Big\{e^{\sum_{k=n}^{n+m-1}t_{k(r)}}\Big\}d(x_n,p)+\Big\{e^{\sum_{k=n}^{n+m-1}t_{k(r)}}\Big\}\Big(\sum_{k=n}^{n+m-1}A_{k(r)}\Big). \quad (21)
\end{aligned}$$

Let $M=e^{\sum_{k=n}^{n+m-1}t_{k(r)}}$. Then $0<M<\infty$ and

$$d(x_{n+m},p) \leq Md(x_n,p)+M\Big(\sum_{k=n}^{n+m-1}A_{k(r)}\Big), \quad (22)$$

for the natural numbers m,n and $p \in F$. Since $\lim_{n\to\infty} d(x_n,F)=0$, therefore for any $\varepsilon>0$, there exists a natural number n_0 such that $d(x_n,F)<\varepsilon/8M$ and $\sum_{k=n}^{n+m-1}A_{k(r)}<\varepsilon/4M$ for all $n \geq n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0},p^*)<\varepsilon/4M$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}
d(x_{n+m},x_n) &\leq d(x_{n+m},p^*)+d(x_n,p^*) \\
&\leq Md(x_{n_0},p^*)+M\sum_{k=n_0}^{\infty}A_{k(r)} \\
&\quad +Md(x_{n_0},p^*)+M\sum_{k=n_0}^{\infty}A_{k(r)} \\
&= 2M\Big(d(x_{n_0},p^*)+\sum_{k=n_0}^{\infty}A_{k(r)}\Big) \\
&\leq 2M\Big(\frac{\varepsilon}{4M}+\frac{\varepsilon}{4M}\Big)=\varepsilon. \quad (23)
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since K is closed, therefore $z \in K$. Next, we show that $z \in F$. Now, the following two inequalities:

$$
\begin{aligned}
d(z,p) &\leq d(z,x_n)+d(x_n,p), \quad \forall p \in F,\ n \geq 1,\\
d(z,x_n) &\leq d(z,p)+d(x_n,p), \quad \forall p \in F,\ n \geq 1,
\end{aligned}
$$

give

$$
-d(z,x_n) \leq d(z,F)-d(x_n,F) \leq d(z,x_n),\ n \geq 1. \tag{24}
$$

That is,

$$
|d(z,F)-d(x_n,F)| \leq d(z,x_n),\ n \geq 1. \tag{25}
$$

As $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n,F)=0$, we conclude that $z \in F$, that is, $\{x_n\}$ converges strongly to a point in F. This completes the proof.

We deduce some results from Theorem 3 as follows.

Corollary 4. *Let X be a complete CAT(0) space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-maps on K with sequences $\{u_{n(i)}\}, \{s_{n(i)}\} \subset [0,\infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty} u_{n(i)} < \infty$ and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$. Assume that $F = \cap_{i=1}^{r} F(T_i)$ is closed. Let $\{x_n\}$ be the general iteration scheme defined by (14). Then the sequence $\{x_n\}$ converges strongly to a point p in F if and only there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a point $p \in F$.*

Corollary 5. *Let X be a complete CAT(0) space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive self-maps on K with sequences $\{u_{n(i)}\} \subset [0,\infty)$ for each $i \in I$ such that $\sum_{n=1}^{\infty} u_{n(i)} < \infty$. Assume that $F = \cap_{i=1}^{r} F(T_i)$ is closed. Let $\{x_n\}$ be the general iteration scheme defined by (14). Then the sequence $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n,F)=0$.*

Proof. The proof of Corollary 5 follows from Theorem 3 with $s_{n(i)}=0$ for each $i \in I$ and for all $n \geq 1$. This completes the proof.

Corollary 6. *Let X be a Banach space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive self-maps on K with sequences $\{u_{n(i)}\} \subset [0,\infty)$ for each $i \in I$ such that $\sum_{n=1}^{\infty} u_{n(i)} < \infty$.*

Assume that $F = \cap_{i=1}^{r} F(T_i)$ is closed. Let $\{x_n\}$ be the general iteration scheme defined by (14). Then the sequence $\{x_n\}$ is of monotone type (A) and monotone type (B) with respect to F. Moreover, $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The proof of Corollary 6 follows from Corollary 5 by setting $\lambda x \oplus (1-\lambda)y = \lambda x + (1-\lambda)y$. This completes the proof.

Theorem 7. *Let X be a complete $CAT(0)$ space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-maps on K with sequences $\{u_{n(i)}\}, \{s_{n(i)}\} \subset [0,\infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty} u_{n(i)} < \infty$ and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$. Assume that $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the general iteration scheme defined by (14) with $0 < \delta \leq \alpha_{n(i)} < 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges to $p \in F$ provided $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, for each $i \in I$ and one member of the family $\{T_i : i \in I\}$ is semi-compact.*

Proof. Without loss of generality, we assume that T_1 is semi-compact. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in K$. Hence, for any $i \in I$, we have

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i q) \\ &\leq (1+L) d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) \to 0. \end{aligned}$$

Thus $q \in F$. By Lemma 2 and Theorem 3, $x_n \to q$. This shows that $\{x_n\}$ converges to a point in F. This completes the proof.

Theorem 8. *Let X be a complete $CAT(0)$ space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-maps on K with sequences $\{u_{n(i)}\}, \{s_{n(i)}\} \subset [0,\infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty} u_{n(i)} < \infty$ and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$. Assume that $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the general iteration scheme defined by (14) with $0 < \delta \leq \alpha_{n(i)} < 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. Suppose that the mappings $\{T_i : i \in I\}$ for each $i \in I$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i \in I$;

(ii) there exists a constant $A > 0$ such that $d(x_n, T_i x_n) \geq A d(x_n, F)$ for each $i \in I$ and for all $n \geq 1$.

Then $\{x_n\}$ converges strongly to a point in F.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 3, that $\{x_n\}$ must converges strongly to a point in F. This completes the proof.

Remark 1. *(1) The approximation result about*

(i) modified Mann iterations in [9] in Hilbert spaces,

(ii) modified Ishikawa iterations in Banach spaces [1, 5, 6], and

(iii) the three-step iteration scheme in uniformly convex Banach spaces from [4, 8, 11] are immediate consequences of our results.

(2) Our results also extend the results of Khan et al. [24] to the case of more general class of asymptotically quasi-nonexpansive mappings consider in this paper.

(3) Theorem 7 generalizes Theorem 3.2 of Xiao et al. [10] in the setup of $CAT(0)$ *spaces.*

(4) Our results also generalize the results of [3] in the setup of $CAT(0)$ *spaces.*

Remark 2. *Any CAT*(k) *space is a CAT*(k') *space for every* $k' \geq k$ *(see [16], p.165), therefore the results in this paper can be applied to any CAT*(k) *space with* $k \leq 0$.

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