Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 39/2014
pp. 281-291
doi: 10.17114/j.aua.2014.39.23

DISCRETE CHARACTERIZATION OF EXPONENTIAL STABILITY OF EVOLUTION FAMILY OVER HILBERT SPACE

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Abstract. In this article we prove that if $\mathcal{U} = \{U(m,n)\}_{m\geq n\geq 0}$ is a positive q-periodic discrete evolution family of bounded linear operators acting on a complex Hilbert space H then $\mathcal{U}$ is uniformly exponentially stable if for each unit vector x in H the series $\sum_{m=0}^{\infty}\phi(|\langle U(m,0)x,x\rangle|)$ is bounded, where $\phi:\mathbb{R}_+ := [0,\infty)\to\mathbb{R}_+$ is a non decreasing function such that $\phi(0)=0$ and $\phi(t)>0$ for all $t\in(0,\infty)$. We also prove the converse of the above result by putting an extra condition i.e. if $\mathcal{U}$ is uniformly exponentially stable and $\sum_{i=0}^{\infty}\phi(x_i)=\phi(\sum_{i=0}^{\infty}(x_i))$ for any $x_i\in\mathbb{R}_+$ then the series $\sum_{m=0}^{\infty}\phi(|\langle U(m,0)x,x\rangle|)$ is bounded.

2000 *Mathematics Subject Classification*: 35B35.

Keywords: Semigroup, discrete evolution family, uniform exponential stability.

1. Introduction

In 1970, Datko [7] brought forth one of the remarkable result in the stability of strongly continuous semigroup which argues that a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators acting on complex or real Banach space is uniformly exponentially stable if and only if

$$\int_0^\infty \|T(t)x\|dt < \infty.$$

In 1972, Pazy [13] had a research on the results of Datko and further improved his attempt by stating that a strongly continuous semigroup of bounded linear operators acting on real or complex Banach space is uniformity exponentially stable if and only if

$$\int_0^\infty \|T(t)x\|^p dt < \infty, \text{ for any } p\geq 1.$$

Rolewicz [17] generalizes the Pazy theorem a step ahead. He state that if

$$\int_0^\infty \phi\|T(t)x\|dt < \infty,$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t \in (0, \infty)$, onward we will call this function as an $\mathcal{R}$-function. Later on, special cases were proved by Zabcyzyk and Przyluski, details can be found in [21] and [16] respectively. Zheng [23] and W. Littman [11] obtained the new proofs of Rolewicz from which they discard the condition of continuity on ϕ.

Let X be a Banach space and X^* be its dual space, then $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is called weak L^p stable for $p \geq 1$, if

$$\int_0^\infty |\langle T(t)x, x^*\rangle|^p dt < \infty. \tag{1}$$

The weak L^p stability of a semi group does not imply uniform exponential stability, counter examples can be found in [8, 9, 12]. For further results on this topic we recommend, [1, 2, 3, 18, 20].

Recently C. Buse and G. Rahmat [5] tried to extend the result given in (1) to evolution family by Weak Rolewicz type approach i.e. they proved that if

$$\int_0^\infty \phi(|\langle U(t,0)x, x\rangle|)dt < \infty,$$

where ϕ is a function as defined above then $\mathcal{U}$ is uniformly exponentially stable.

Our paper is the continuation of the last coated paper in discrete form. Different simultaneous results concerning discrete semigroups and discrete evolution family can be found in [4, 6, 10, 14, 15, 19, 22].

In the first section of this article we will give some preliminaries and in second section we will present our main results.

2. Notations and Preliminaries

We denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_+$ the sets of real numbers, complex numbers and positive integers respectively. $\sigma(A)$ denotes the spectral radius of A. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators acting on X. As usual $\langle\cdot,\cdot\rangle$ denotes the scalar product on a Hilbert space H. The norms in $X, H, \mathcal{L}(X)$ and $\mathcal{L}(H)$ will be denoted by the same symbol, namely $\|\cdot\|$.

The family $\mathcal{U} = \{U(m,n) : m, n \in \mathbb{Z}_+, m \geq n\}$ is called q-periodic discrete evolution family if it satisfies the following properties.

- (i) $U(m,m) = I$.
- (ii) $U(m,n)U(n,r) = U(m,r)$.

- (iii) $U(m+q, n+q) = U(m,n)$.

It is well known that $\mathcal{U}$ is exponentially bounded, that is, there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$\|U(m,n)\| \leq M_\omega e^{\omega(m-n)}, \quad \text{for all} \quad m \geq n. \tag{2}$$

The growth bound of exponentially bounded evolution family $\mathcal{U}$ is defined by

$$\omega_0(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \text{ there is } M_\omega \geq 0 \text{ such that (2) holds}\}.$$

A bounded linear operator A, acting on a Hilbert space H, is positive if $\langle Ax, x\rangle \geq 0$ for every $x \in H$. An evolution family $\{U(m,n) : m \geq n\}$ is called self-adjoint (positive) if each operator $U(m,n)$ with $m \geq n$, is self-adjoint (respectively positive). The family $\mathcal{U}$ is uniformly exponentially stable if its growth bound is negative. An evolution family is self-adjoint if each member of the family is self-adjoint.

Here we will recall few lemmas from [5], without proof, so that the paper will be self-contained.

Lemma 1 ([5]). *Let X be a complex Banach space and let $V \in \mathcal{L}(X)$. If the spectral radius of V is greater or equal to 1, then for all $0 < \varepsilon < 1$ and any sequence (a_n) with $a_n \to 0$ (as $n \to \infty$) and $\|(a_n)\|_\infty \leq 1$, there exists a unit vector $u_0 \in X$, such that*

$$\|V^n u_0\| \geq (1-\varepsilon)|a_n|, \quad \textit{for all } n \in \mathbb{Z}_+.$$

Throughout this article, (t_n) will be a sequence of nonnegative real numbers, such that $1 \leq q \leq t_{n+1} - t_n \leq \alpha$ for every $n \in \mathbb{Z}_+$ and some positive real number α.

Lemma 2 ([5]). *Let $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be a strongly continuous q-periodic ($q \geq 1$) evolution family of bounded linear operators acting on a Banach space X and let (t_n) be a sequence as given before. If the evolution family is not uniformly exponentially stable, then there exists a positive constant C, having the properties: for every $\mathbb{C}$-valued sequence (b_n) with $b_n \to 0$ (as $n \to \infty$) and $\|(b_n)\|_\infty \leq 1$, there exists a unit vector $u_0 \in X$, such that*

$$\|U(t_n, 0)u_0\| \geq C|b_{n+1}|, \quad \textit{for all} \quad n \in \mathbb{Z}_+. \tag{3}$$

An evolution family $\mathcal{U}$ is said to satisfy the weak discrete Rolewicz condition, equation (2.4) in [5], if

$$\sum_{n=0}^{\infty} \phi(|\langle U(t_n, 0)x, y\rangle|) < \infty. \tag{4}$$

Lemma 3 ([5])**.** *Let ϕ be an $\mathcal{R}$-function and let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous q-periodic ($q \geq 1$) evolution family acting on a Banach space X. If the family $\mathcal{U}$ satisfies*

$$\sum_{n=0}^{\infty} \phi(\|U(t_n,0)x\|) < \infty,$$

then it is uniformly exponentially stable.

3. Main Results

Let $\mathbf{T}$ be a discrete semigroup of bounded linear operators acting on complex Hilbert space H. When $\mathbf{T}$ is self-adjoint i.e. $T(m) = T^*(m)$, for every $m \in \mathbb{Z}_+$ then

$$\langle T(m)x, x\rangle = \langle T(m/2)T(m/2)x, x\rangle = \|T(m/2)x\|^2.$$

Let K and L be two self-adjoint operators then we have the following inequality

$$||\langle KLx, y\rangle||^2 \leq \langle K^2y, y\rangle\langle L^2x, x\rangle \quad \text{for all} \quad x,\ y \in H. \tag{5}$$

We are in the position to state our first theorem.

Theorem 4. *Let ϕ be an $\mathcal{R}$-function and let $\mathbf{T} = \{T(m)\}_{m\in\mathbb{Z}_+}$ be a self-adjoint discrete semigroup of bounded linear operators acting on a complex Hilbert space H. Then*

1. *If the series $\sum_{m=0}^{\infty} \phi(|\langle T(m)x, x\rangle|)$ is bounded for all $x \in H$ with $\|x\| = 1$ then $\mathbf{T}$ is uniformly exponentially stable.*
2. *If the semigroup $\mathbf{T}$ is uniformly exponentially stable and $\sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty}(x_i))$ for any $x_i \in \mathbb{R}_+$ then the series $\sum_{m=0}^{\infty} \phi(|\langle T(m)x, x\rangle|)$ is bounded.*

Proof. Case 1. Using inequality (5), we can write

$$\begin{aligned}
|\langle T(2m)x, y\rangle|^2 &= |\langle T(2m-n)T(n)x, y\rangle|^2\\
&\leq \langle T(4m-2n)y, y\rangle\langle T(2n)x, x\rangle\\
&\leq Me^{8\omega}\langle T(2n)x, x\rangle.
\end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$||\langle T(2m)x, x\rangle||^2 \leq Me^{8w}\langle T(2n)x, x\rangle.$$

As ϕ is an increasing function, so we can write

$$\phi(1/Me^{8w}\|T(m)x\|^4) \leq \phi(|\langle T(2n)x, x\rangle|).$$

Taking summation on both sides

$$\sum_{n=0}^{\infty}\phi(1/Me^{8w}\|T(m)x\|^4) \leq \sum_{n=0}^{\infty}\phi(|\langle T(2n)x,x\rangle|).$$

Since

$$\sum_{n=0}^{\infty}\phi(|\langle T(2n)x,x\rangle|) < \infty,$$

so

$$\sum_{n=0}^{\infty}\phi\Big(\frac{1}{Me^{8w}}\|T(m)x\|^4\Big) < \infty.$$

Hence by using Lemma 3, we can say that **T** is uniformly exponentially stable.

Case 2. Let **T** is uniformly exponentially stable, then there exists $v > 0$ and $M \geq 0$ such that

$$\|T(m)\| \leq Me^{-vm},$$

replacing m by m/2

$$\|T(m/2)x\| \leq Me^{-vm/2}$$

or

$$\sqrt{\langle T(m)x,x\rangle} \leq Me^{-vm/2}.$$

applying ϕ

$$\phi(\langle T(m)x,x\rangle) \leq \phi(M^2e^{-vm}).$$

Taking summation on both sides

$$\begin{aligned}\sum_{m=0}^{\infty}\phi(\langle T(m)x,x\rangle) &\leq \sum_{m=0}^{\infty}\phi(M^2e^{-vm})\\ &= \phi(\sum_{m=0}^{\infty}M^2e^{-vm})\\ &\leq \phi(\frac{M^2e^v}{e^v-1}).\end{aligned}$$

By definition of ϕ, $\phi(\frac{M^2e^v}{e^v-1}) < \infty$. Hence

$$\sum_{m=0}^{\infty}\phi(\langle T(m)x,x\rangle) < \infty.$$

Thus proof is complete.

We also extend the above result to self-adjoint q-discrete evolution family $\mathcal{U}$ as follows.

Theorem 5. *Let $\mathcal{U} = \{U(m,n) : m,n \in \mathbb{Z}_+,\ m \geq n\}$ is a self-adjoint q-periodic discrete evolution family acting on a complex Hilbert space H then the following two statement holds true*

1. *If the series $\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|)$ is bounded for all $x \in H$ with $\|x\| = 1$ then $\mathcal{U}$ is uniformly exponentially stable.*
2. *If the evolution family $\mathcal{U}$ is uniformly exponentially stable and $\sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty}(x_i))$ for any $x_i \in \mathbb{R}_+$ then the series $\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|)$ is bounded.*

Proof. Case 1. Using inequality (5) we can write

$$\begin{aligned} |\langle U(m,0)x, y\rangle|^2 &= |\langle U(m,n)U(n,0)x, y\rangle|^2 \\ &\leq \langle U^2(m,n)y, y\rangle\langle U^2(n,0)x, x\rangle \\ |\langle U(m,0)x, y\rangle|^2 &\leq \|U(m,n)y\|^2\|U(n,0)x\|^2 \\ &\leq M^2e^{4qw}\|U(n,0)x\|^2. \end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$1/Me^{2qw}|\langle U(m,0)x, x\rangle| \leq \|U(n,0)x\|.$$

Since ϕ is an increasing function, so we can write

$$\phi(1/Me^{2qw}|\langle U(m,0)x, x\rangle|) \leq \phi(\|U(n,0)x\|).$$

Taking summation on both sides

$$\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x, x\rangle|) \leq \sum_{n=0}^{\infty} \phi(\|U(n,0)x\|).$$

Since

$$\sum_{n=0}^{\infty} \phi(\|U(n,0)x\|) < \infty,$$

so

$$\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x, x|\rangle) < \infty.$$

Equivalently, we can write

$$\sum_{m=0}^{\infty} \phi(1/Me^{2qw}\|U(m,0)x\|^2) < \infty.$$

So again using Lemma 3, we can say that $\mathcal{U}$ is uniformly exponentially stable.

Case 2. Let $\mathcal{U}$ is uniformly exponentially stable, then there exists two positive constants v and M such that

$$\|U(m,n)\| \leq Me^{-v(m-n)}.$$

Putting $n = 0$, we get

$$\|U(m,0)\| \leq Me^{-vm}.$$

Since ϕ is an increasing function, so we can write

$$\phi(\|U(m,0)x\|) \leq \phi(Me^{-vm}).$$

Taking summation on both sides

$$\begin{aligned}\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) &\leq \sum_{m=0}^{\infty} \phi(Me^{-vm}) \\ &= \phi(\sum_{m=0}^{\infty} Me^{-vm}) \\ &\leq \phi(\frac{Me^{v}}{e^{v}-1}).\end{aligned}$$

By definition of ϕ we have

$$\phi(\frac{Me^{v}}{e^{v}-1}) < \infty,$$

hence

$$\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) < \infty.$$

Thus the series $\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|)$ is bounded.

We also extend similar idea to positive discrete evolution family.

Theorem 6. *Let $\mathcal{U} = \{U(m,n) : m, n \in \mathbb{Z}_+\}$ is a positive q-periodic discrete evolution family acting on a complex Hilbert space H then the following two statement holds true*

1. *If the series $\sum_{m=0}^{\infty} \phi(\langle U(m,0)x, x\rangle)$ is bounded for all $x \in H$ with $\|x\| = 1$ then $\mathcal{U}$ is uniformly exponentially stable.*
2. *If the evolution family $\mathcal{U}$ is uniformly exponentially stable and $\sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty}(x_i))$ for any $x_i \in \mathbb{R}_+$ then the series $\sum_{m=0}^{\infty} \phi(\langle U(m,0)x, x\rangle)$ is bounded.*

Proof. Case 1. Using inequality (5) we can write

$$\begin{aligned}|\langle U^{1/2}(m,0)x,y\rangle|^2 &= |\langle U^{1/2}(m,n)U^{1/2}(n,0)x,y\rangle|^2\\ &\leq \langle U(m,n)y,y\rangle\langle U(n,0)x,x\rangle\\ |\langle U^{1/2}(m,0)x,y\rangle|^2 &\leq \langle U(m,n)y,y\rangle\langle U(n,0)x,x\rangle\\ &\leq Me^{2qw}\langle U(n,0)x,x\rangle.\end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$1/Me^{2qw}|\langle U(m,0)x,x\rangle|^2 \leq \langle U(n,0)x,x\rangle.$$

Since ϕ is an increasing function, so we can write

$$\phi(1/Me^{2qw}|\langle U(m,0)x,x\rangle|^2) \leq \phi(\langle U(n,0)x,x\rangle).$$

Taking summation on both sides

$$\sum_{m=0}^{\infty}\phi(1/Me^{2qw}|\langle U(m,0)x,x\rangle|^2) \leq \sum_{n=0}^{\infty}\phi(\langle U(n,0)x,x\rangle).$$

Since

$$\sum_{n=0}^{\infty}\phi(\langle U(n,0)x,x\rangle) < \infty,$$

so

$$\sum_{m=0}^{\infty}\phi(1/Me^{2qw}|\langle U(m,0)x,x\rangle|^2) < \infty.$$

Hence using Lemma 3, we can say that $\mathcal{U}$ is uniformly exponentially stable.

Case 2. Let $\mathcal{U}$ is uniformly exponentially stable, then there exists $v \in \mathbb{R}$ and $M \geq 0$ such that

$$\|U(m,n)\| \leq Me^{-v(m-n)},$$

Putting $n = 0$ we get

$$\|U(m,0)\| \leq Me^{-vm},$$

replacing m by $\frac{m}{2}$

$$\|U(m/2,0)x\| \leq Me^{-vm/2}$$

$$\sqrt{\langle U(m,0)x,x\rangle} \leq Me^{-vm/2}.$$

Since ϕ is increasing function so we can write

$$\phi(\langle U(m,0)x,x\rangle) \leq \phi(M^2e^{-vm}).$$

Taking summation on both sides

$$\begin{aligned}\sum_{m=0}^{\infty}\phi(\langle U(m,0)x,x\rangle) &\leq \sum_{m=0}^{\infty}\phi(M^2e^{-vm})\\ &= \phi(\sum_{m=0}^{\infty}M^2e^{-vm})\\ &= \phi(\sum_{m=0}^{\infty}\frac{M^2}{e^{vm}})\\ &\leq \phi(\frac{M^2e^v}{e^v-1})<\infty.\end{aligned}$$

Hence

$$\sum_{m=0}^{\infty}\phi(\langle U(m,0)x,x\rangle)<\infty.$$

Which completes the proof.

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