Acta Universitatis Apulensis No. 39/2014
ISSN: 1582-5329 pp. 249-261
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.39.20

SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY MODIFIED CATA'S OPERATOR

A.O. Mostafa, M.K. Aouf, A. Shamandy and E.A. Adwan

Abstract. In this paper, we introduce a new class of harmonic univalent functions defined by modified Cata's operator. Coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class are obtained and also for a class preserving integral operator.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: Harmonic univalent functions, Cata's operator, extreme points.

1. Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [5]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.2}$$

In [5] Clunie and Shell-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

Let $\overline{S_H}$ denote the subclasses of S_H consisting of functions $f = h + \overline{g}$ such that h and g given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ g(z) = (-1)^m \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.3}$$

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}, \mu \geq 0$ and $l \geq 0$, the extended multiplier transformation $I^m(\mu, l)$ is defined by the following infinite series (see [2]):

$$I^m(\mu, l)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l + \mu(n-1)}{1 + l}\right]^m a_n z^n. \tag{1.4}$$

It follows from (1.4) that (see [2])

$$\mu z(I^m(\mu, l)f(z))' = (1 + l)I^{m+1}(\mu, l)f(z) - (1 - \mu + l)I^m(\mu, l)f(z) \ (\mu > 0)$$

and

$$I^{m_1}(\mu, l)(I^{m_2}(\mu, l)f(z)) = I^{m_1+m_2}(\mu, l)f(z) = I^{m_2}(\mu, l)(I^{m_1}(\mu, l)f(z)),$$

for all integers m_1 and m_2.
We note that:

$$I^0(\mu, l)f(z) = f(z) \ and \ I^1(1, 0)f(z) = zf'(z).$$

Also, we can write

$$I^m(\mu, l)f(z) = (\Phi^m_{\mu,l} * f)(z),$$

where

$$\Phi^m_{\mu,l}(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l + \mu(n-1)}{1 + l}\right]^m z^n.$$

Now we can define the modified Cata's operator as follows:

$$I(m, \mu, l) f(z) = I^m(\mu, l) h(z) + (-1)^m \overline{I^m(\mu, l) g(z)}, \tag{1.5}$$

where

$$I^m(\mu, l) h(z) = z + \sum_{n=2}^{\infty} \left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m a_n z^n$$

and

$$I^m(\mu,l)\,g(z)=(-1)^m\sum_{n=1}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m b_n z^n.$$

For $1<\gamma\leq 2$ and for all $z\in U$, let $SI^m(\mu,l;\gamma)$ denote the family of harmonic functions $f(z)=h+\overline{g}$, where h and g given by (1.2) and satisfying the analytic criterion

$$Re\left\{\frac{I^m(\mu,l)\,h(z)+(-1)^m\,\overline{I^m(\mu,l)\,g(z)}}{z}\right\}<\gamma. \tag{1.6}$$

Let $\overline{SI^m}(\mu,l;\gamma)$ be the subclass of $SI^m(\mu,l;\gamma)$ consisting of functions $f=h+\overline{g}$ such that h and g given by (1.3).

We note that for suitable choices of m,μ and l, we obtain the following subclasses:

(1) Putting $\mu=1$ and $l=0$, in (1.6), the class $\overline{SI^m}(1,0;\gamma)$ reduces to the class $\overline{SI^m}(\gamma)$

$$=\left\{f\in\ S_H:Re\left\{\frac{D^m h(z)+(-1)^m\,\overline{D^m g(z)}}{z}\right\}<\gamma,\ 1<\gamma\leq 2, m\in\mathbb{N}_0\right.$$
$$\left. z\in U\right\},$$

where D^m is the modified Salagean operator (see [7]), the differential opertor D^m was introduced by Salagean (see [8]);

(2) Putting $\mu=1$ and $l=1$, in (1.6), the class $\overline{SI^m}(1,1;\gamma)$ reduces to the class reduces to the class $\overline{SI^m}(\gamma)$

$$=\left\{f\in S_H:Re\left\{\frac{I^m h(z)+(-1)^m\,\overline{I^m g(z)}}{z}\right\}<\gamma\ ,1<\gamma\leq 2, m\in\mathbb{Z}=\{0,\pm1,\right.$$
$$\left.\pm2,...\}\,,z\in U\right\},$$

where I^m is the modified Uralegaddi-Somanatha operator (see [9]), defined as follows:

$$I^m f(z)=I^m h(z)+(-1)^m\,\overline{I^m g(z)};$$

(3) Putting $\mu=1$, in(1.6), the class $\overline{SI^m}(1,l;\gamma)$ reduces to the class $\overline{SI^m}(l;\gamma)$

$$=\left\{f\in S_H:Re\left\{\frac{I_l^m h(z)+(-1)^m\,\overline{I_l^m g(z)}}{z}\right\}<\gamma,\ 1<\gamma\leq 2, m\in\mathbb{R}, l>-1, z\in U\right\},$$

where I_l^m is the modified Cho-Kim operator [3] (also see [4]), defined as follows:

$$I_l^m f(z) = I_l^m h(z) + (-1)^m \overline{I_l^m g(z)};$$

(4) Putting $l = 0$, in(1.6), the class $\overline{SI^m}(\mu, 0; \gamma)$ reduces to the class $\overline{SI^m}(\mu; \gamma)$

$$= \left\{ f \in S_H : Re \left\{ \frac{D_\mu^m h(z) + (-1)^m \overline{D_\mu^m g(z)}}{z} \right\} < \gamma, \ 1 < \gamma \leq 2, \mu \geq 0, m \in \mathbb{N}_0, z \in U \right\},$$

where D_μ^m is the modified Al-Oboudi operator (see [1]), defined as follows:

$$D_\mu^m f(z) = D_\mu^m h(z) + (-1)^m \overline{D_\mu^m g(z)}.$$

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $1 < \gamma \leq 2, \ m \in \mathbb{N}_0, \mu \geq 0$ and $l \geq 0$.

Theorem 1. *Let $f = h + \overline{g}$ be so that $h(z)$ and $g(z)$ given by (1.2). Furthermore, let*

$$\sum_{n=2}^{\infty} \left[\frac{1 + l + \mu(n-1)}{1+l} \right]^m |a_n| + \sum_{n=1}^{\infty} \left[\frac{1 + l + \mu(n-1)}{1+l} \right]^m |b_n| \leq \gamma - 1. \tag{2.1}$$

Then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z) \in SI^m(\mu, l; \gamma)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum\limits_{n=1}^{\infty} b_n (z_1^n - z_2^n)}{(z_1^n - z_2^n) + \sum\limits_{n=1}^{\infty} a_n (z_1^n - z_2^n)} \right| \\ &> 1 - \frac{\sum\limits_{n=1}^{\infty} n |b_n|}{1 - \sum\limits_{n=2}^{\infty} n |a_n|} \geq 1 - \frac{\frac{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m}{\gamma-1} |b_n|}{\frac{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m}{\gamma-1} |a_n|} \geq 0, \end{aligned}$$

which proves univalence. Note that $f(z)$ is sense-preserving in U. This is because

$$
\begin{aligned}
\left|h^{'}(z)\right| &\geq 1-\sum_{n=2}^{\infty} n\left|a_n\right|\left|z^{n-1}\right| \\
&> 1-\sum_{n=2}^{\infty} n\left|a_n\right| \geq 1-\sum_{n=2}^{\infty} \frac{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m}{\gamma-1}\left|a_n\right| \\
&\geq \sum_{n=1}^{\infty} \frac{\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m}{\gamma-1}\left|b_n\right| \geq \sum_{n=1}^{\infty} n\left|b_n\right| \\
&> \sum_{n=1}^{\infty} n\left|b_n\right|\left|z^{n-1}\right| \geq\left|g^{'}(z)\right| .
\end{aligned}
$$

Now we will show that $f(z) \in SI^m(\mu, l; \gamma)$. We only need to show that if (2.1) holds then the condition (1.6) is satisfied.
Using the fact that $Re\{w\} < \gamma$ if and only if $|w-1| < |w-(2\gamma-1)|$, it suffices to show that

$$
\left|\frac{\frac{I^m(\mu,l)\,h(z)+\overline{I^m(\mu,l)\,g(z)}}{z}-1}{\frac{I^m(\mu,l)\,h(z)+\overline{I^m(\mu,l)\,g(z)}}{z}-(2\gamma-1)}\right| < 1.
$$

We have

$$
\begin{aligned}
&\left|\frac{\frac{I^m(\mu,l)\,h(z)+\overline{I^m(\mu,l)\,g(z)}}{z}-1}{\frac{I^m(\mu,l)\,h(z)+\overline{I^m(\mu,l)\,g(z)}}{z}-(2\gamma-1)}\right| \\
= &\left|\frac{\sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m a_n z^{n-1}+\sum\limits_{n=1}^{\infty}(-1)^m\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m \overline{b_n z^{n-1}}}{2(\gamma-1)+\sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m a_n z^{n-1}+\sum\limits_{n=1}^{\infty}(-1)^m\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m \overline{b_n z^{n-1}}}\right|
\end{aligned}
$$

$$\leq \frac{\sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n|\left|z^{n-1}\right| + \sum\limits_{n=1}^{\infty}(-1)^{m}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|\left|z^{n-1}\right|}{2(\gamma-1) - \sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n|\left|z^{n-1}\right| - \sum\limits_{n=1}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|\left|z^{n-1}\right|}$$

$$< \frac{\sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n| + \sum\limits_{n=1}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|}{2(\gamma-1) - \sum\limits_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n| - \sum\limits_{n=1}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|} \leq 1$$

which is bounded above by 1 by using (2.1). This completes the proof of Theorem 1.

Theorem 2. *A function $f(z)$ of the form (1.1) is in the class $\overline{SI^m}(\mu, l; \gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n| + \sum_{n=1}^{\infty}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n| \leq \gamma - 1.$$

Proof. Since $\overline{SI^m}(\mu, l; \gamma) \subset SI^m(\mu, l; \gamma)$, we only need to prove the "only if" part of this theorem. To this end , for functions $f(z)$ of the form (1.3), we notice that the condition

$$Re\left\{\frac{I^{m}(\mu,l)\,h(z) + (-1)^{m}\,\overline{I^{m}(\mu,l)\,g(z)}}{z}\right\} < \gamma$$

is equivalent to

$$Re\left\{1 + \sum_{n=2}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n|\,z^{n-1} + \sum_{n=1}^{\infty}(-1)^{m}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|\,\overline{z^{n-1}}\right\}$$

$$\leq 1 + \sum_{n=2}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^{m}|a_n|\left|z^{n-1}\right| + \sum_{n=1}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^{m}|b_n|\left|\overline{z^{n-1}}\right| < \gamma.$$

Letting $z \to 1^{-}$ along the real axis, we obtain the inequality (2.1). This completes the proof of Theorem 2.

Remark 1. *Putting $\mu = 1, l = 0$ and $m = 1$ in Theorem 2, we obtain the result obtained by Dixit and Porwal [6, Theorem 2.1].*

3. Distortion theorem

Theorem 3. *Let the function $f(z)$ defined by (1.1) belong to the class $SI^m(\mu,l;\gamma)$. Then for $|z|=r<1$,we have*

$$(1-|b_1|)\,r-\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,r^2\leq|f(z)|$$

$$\leq(1+|b_1|)\,r+\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,r^2 \tag{3.1}$$

for $|b_1|\leq\gamma-1$. The results are sharp with equality for the functions $f(z)$ defined by

$$f(z)=z+b_1\overline{z}-\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,\overline{z}^2 \tag{3.2}$$

and

$$f(z)=z-b_1\overline{z}-\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,z^2. \tag{3.3}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z)\in\overline{SI^m}(\mu,l;\gamma)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n\leq(1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^2\\
&= (1+|b_1|)r+\frac{(\gamma-1)\,(1+l)^m}{(1+l+\mu)^m}\sum_{n=2}^{\infty}\left(\tfrac{(1+l+\mu)^m}{(\gamma-1)(1+l)^m}|a_n|+\tfrac{(1+l+\mu)^m}{(\gamma-1)(1+l)^m}|b_n|\right)r^2\\
&\leq (1+|b_1|)r+\tfrac{(\gamma-1)(1+l)^m}{(1+l+\mu)^m}\sum_{n=2}^{\infty}\left[\tfrac{[1+l+\mu(n-1)]^m}{(\gamma-1)(1+l)^m}|a_n|+\tfrac{[1+l+\mu(n-1)]^m}{(\gamma-1)(1+l)^m}|b_n|\right]r^2\\
&\leq (1+|b_1|)r+\tfrac{(\gamma-1)(1+l)^m}{(1+l+\mu)^m}\left(1-\frac{|b_1|}{\gamma-1}\right)r^2\\
&= (1+|b_1|)r+\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,r^2.
\end{aligned}$$

Similarly we can prove $|f(z)|\geq(1-|b_1|)\,r-\left[\frac{1+l}{1+l+\mu}\right]^m(\gamma-1-|b_1|)\,r^2$. The functions $f(z)$ given by (3.2) and (3.3), respectively, for $|b_1|\leq\gamma-1$ show that the bounds given in Theorem 3 are sharp.

4. Extreme points

Theorem 4. *Let $f(z)$ be given by (1.1). Then $f(z) \in \overline{SI^m}(\mu, l; \gamma)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} \left(\mu_n h_n(z) + \eta_n g_n(z)\right), \tag{4.1}$$

where $h_1(z) = z$,

$$h_n(z) = z + \left[\frac{1+l}{1+l+\mu(n-1)}\right]^m (\gamma - 1) z^n \qquad (n \geq 2; m \in \mathbb{N}_0) \tag{4.2}$$

and

$$g_n(z) = z + (-1)^m \left[\frac{1+l}{1+l+\mu(n-1)}\right]^m (\gamma - 1) \overline{z^n} \qquad (n \geq 1; m \in \mathbb{N}_0), \tag{4.3}$$

$\mu_n \geq 0, \eta_n \geq 0, \sum\limits_{n=1}^{\infty} (\mu_n + \eta_n) = 1$. *In particular, the extreme points of the class $\overline{SI^m}(\mu, l; \gamma)$ are $\{h_n\}$ and $\{g_n\}$, respectively.*

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \left(\mu_n h_n(z) + \eta_n g_n(z)\right) \\ &= z + \sum_{n=2}^{\infty} (\gamma - 1) \left[\frac{1+l}{1+l+\mu(n-1)}\right]^m \mu_n z^n \\ &\quad + (-1)^m \sum_{n=1}^{\infty} (\gamma - 1) \left[\frac{1+l}{1+l+\mu(n-1)}\right]^m \eta_n \overline{z^n}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \tfrac{[1+l+\mu(n-1)]^m}{(\gamma-1)(1+l)^m} \left(\tfrac{(\gamma-1)(1+l)^m}{[1+l+\mu(n-1)]^m} \mu_n\right) + \sum_{n=1}^{\infty} \tfrac{[1+l+\mu(n-1)]^m}{(\gamma-1)(1+l)^m} \left(\tfrac{(\gamma-1)(1+l)^m}{[1+l+\mu(n-1)]^m} \eta_n\right) \\ = &\sum_{n=2}^{\infty} \mu_n + \sum_{n=1}^{\infty} \eta_n = 1 - \mu_1 \leq 1 \end{aligned}$$

and so $f(z) \in \overline{SI^m}(\mu, l; \gamma)$.

Conversely, if $f(z) \in \overline{SI^m}(\mu, l; \gamma)$, then

$$|a_n| \leq (\gamma - 1) \left[\frac{1+l}{1+l+\mu(n-1)}\right]^m$$

and

$$|b_n| \le (\gamma - 1)\left[\frac{1+l}{1+l+\mu(n-1)}\right]^m.$$

Setting

$$\mu_n = \frac{1}{(\gamma-1)}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |a_n| \ \ (n = 2, 3, ...)$$

and

$$\eta_n = \frac{1}{(\gamma-1)}\left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |b_n| \ \ (n = 1, 2, ...).$$

Since $0 \le \mu_n \le 1$ $(n = 2, 3, ...)$ and $0 \le \eta_n \le 1$ $(n = 1, 2, ...)$, $\mu_1 = 1 - \sum\limits_{n=2}^{\infty} \mu_n - \sum\limits_{n=1}^{\infty} \eta_n \ge 0$, then, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of the Theorem 4.

5. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n} \tag{5.1}$$

and

$$F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z^n}, \tag{5.2}$$

the convolution of f and F is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \overline{z^n}. \tag{5.3}$$

Using this definition, the next theorem shows that the class $\overline{SI^m}(\mu, l; \gamma)$ is closed under convolution.

Theorem 5. *For $1 < \gamma \le \lambda \le 2$, let $f \in \overline{SI^m}(\mu, l; \gamma)$ where $f(z)$ is given by (5.1) and $F \in \overline{SI^m}(\mu, l; \lambda)$ where $F(z)$ is given by (5.2). Then $\overline{SI^m}(\mu, l; \gamma) \subset \overline{SI^m}(\mu, l; \lambda)$.*

Proof. We wish to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 1. For $F \in \overline{SI^m}(\mu, l; \lambda)$ we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now, for the convolution function $f * F$ we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |a_n A_n| z^n + \sum_{n=1}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |b_n B_n| \overline{z^n} \\ \leq & \sum_{n=2}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |a_n| z^n + \sum_{n=1}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |b_n| \overline{z^n} \\ \leq & 1, \end{aligned}$$

since $1 < \gamma \leq \lambda \leq 2$ and $f \in \overline{SI^m}(\mu, l; \gamma)$. Therefore $f * F \in \overline{SI^m}(\mu, l; \gamma) \subset \overline{SI^m}(\mu, l; \lambda)$.

Now we show that the class $\overline{SI^m}(\mu, l; \gamma)$ is closed under convex combinations of its members.

Theorem 6. *The class $\overline{SI^m}(\mu, l; \gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...,$ let $f_i \in \overline{SI^m}(\mu, l; \gamma)$, where f_i is given by

$$f_i = z + \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z^n}.$$

Then by using Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |a_{n_i}| z^n + \sum_{n=1}^{\infty} \frac{1}{(\gamma-1)} \left[\frac{1+l+\mu(n-1)}{1+l}\right]^m |b_{n_i}| \overline{z}^n \leq 1. \tag{5.4}$$

For $\sum\limits_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \overline{z^n}. \tag{5.5}$$

Then by (5.4), we have

$$\sum_{n=2}^{\infty} \frac{1}{(\gamma-1)} \left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m \left(\sum_{i=1}^{\infty} t_i \left|a_{n_i}\right|\right) +$$
$$\sum_{n=1}^{\infty} \frac{1}{(\gamma-1)} \left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m \left(\sum_{i=1}^{\infty} t_i \left|b_{n_i}\right|\right)$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{1}{(\gamma-1)} \left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m \left|a_{n_i}\right| + \sum_{n=1}^{\infty} \frac{1}{(\gamma-1)} \left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m \left|b_{n_i}\right|\right)$$
$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by (2.1) and so $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in S_{H_{q,s}}([\alpha_1];\gamma)$.

6. A family of integral operators

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $\overline{SI^m}(\mu,l;\gamma)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)\, dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)\, dt} \quad (c > -1) \tag{6.1}$$

also belongs to the class $\overline{SI^m}(\mu,l;\gamma)$.

Proof. Let the function $f(z)$ be defined by (1.1). Then from the representation (6.1) of $F(z)$, it follows that

$$F(z) = z + \sum_{n=2}^{\infty} d_n z^n + \sum_{n=1}^{\infty} \zeta_n \overline{z^n},$$

where

$$d_n = \left(\frac{c+1}{c+n}\right) |a_n| \quad \text{and} \quad \zeta_n = \left(\frac{c+1}{c+n}\right) |b_n|.$$

Therefore, we have

$$\begin{aligned}
& \sum_{n=2}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m d_n+\sum_{n=1}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m \zeta_n \\
= & \sum_{n=2}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m\left(\frac{c+1}{c+n}\right)|a_n|++\sum_{n=1}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m\left(\frac{c+1}{c+n}\right)|b_n| \\
\leq & \sum_{n=2}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m|a_n|+\sum_{n=1}^{\infty}\left[\tfrac{1+l+\mu(n-1)}{1+l}\right]^m|b_n| \leq \gamma-1,
\end{aligned}$$

since $f(z) \in \overline{SI^m}(\mu,l;\gamma)$. Hence, by Theorem 1, $F(z) \in \overline{SI^m}(\mu,l;\gamma)$.
This completes the proof of Theorem 7.

Remark 2. *Specializing the parameters l, μ and m, in the above results, we obtain the corresponding results for the corresponding classes defined in the introduction.*

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] A. Catas, *On certain classes of p-valent functions defined by multiplier transformations*. In: Proc. Book of the International Symposium on Geometric Functions Theory and Applications, Istanbul, Turkey, (2007), 241-250.

[3] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., 40 (2003), no. 3, 399-410.

[4] N. E. Cho and H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, 37 (2003), no. 1-2, 39-49.

[5] J. Clunie and T. Shell-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 9, (1984), 3-25.

[6] K. K. Dixit and S. Porwal, *A subclass of harmonic univalent functions with positive cooefficents*, Tamkang J. Math., 41 (2010), no. 3, 261-269.

[7] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, *Salagean type harmonic univalent functions*, Southwest J. Pure Appl. Math., 2 (2002), 77-82.

[8] G. S. Salagean, *Subclasses of univalent functions*, Complex Analysis, Lecture Notes in Math. (Springer-Verlag), 1013 (1983), 362-372.

[9] A. Uralegaddi and C. Somanatha, *Certain classes of univalent functions*, in: H.M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singaporr, New Jersey, London, Hong Kong, 1992, 371-374.

A. O. Mostafa, M. K. Aouf, A. Shamandy and E. A. Adwan
Department of Mathematics, Faculty of Science,
University of Mansoura, Mansoura, Egypt
e-mail: *adelaeg254@yahoo.com (Mostafa), mkaouf127@yahoo.com (Aouf), shamandy16@hotmail.com (Shamandy), eman.a2009@yahoo.com (Adwan)*