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A GENERALIZED CLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH SALAGEAN OPERATORS INVOLVING CONVOLUTIONS

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Abstract. In this paper, we introduce a generalized class $\mathcal{S}_H^i\left(m,n,\phi,\psi;\alpha\right)$, $i\in\{0,1\}$ of harmonic univalent functions. A sufficient coefficient condition for the normalized harmonic function to be in this class is obtained. It is also shown that this coefficient condition is necessary for its subclass $\mathcal{TS}_H^i\left(m,n,\phi,\psi;\alpha\right)$. We further, obtain extreme points, bounds and a covering result for the class $\mathcal{TS}_H^i\left(m,n,\phi,\psi;\alpha\right)$ and show that this class is closed under convolutions and convex combinations. In proving our results certain conditions on the coefficients of ϕ and ψ are considered which lead various well-known results proved earlier.

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1. Introduction

A continuous complex-valued function $f=u+iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D}$, we can write $f=h+\overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)|>|g'(z)|,\quad z\in\mathbb{D}$ (see [3]).

Denote by $\mathcal{S}_H$ the class of function $f=h+\overline{g}$ which are harmonic, univalent and sense-preserving in the open unit disc $\mathbb{U}=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\overline{g}\in\mathcal{S}_H$ we may express the analytic functions h and g as

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\ g(z)=\sum_{k=1}^{\infty}b_kz^k,\quad |b_1|<1. \tag{1}$$

Note that the class $\mathcal{S}_H$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of f i.e. $g \equiv 0$. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

For more basic results on harmonic functions one may refer to the following introductory text book by Duren [7] (see also [1], [12], [13] and the references there in). For $f = h + \overline{g}$ with h and g are of the form (1), Jahangiri et al. [10] defined the modified Salagean operator $\mathcal{D}^n$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by

$$\mathcal{D}^n f(z) = \mathcal{D}^n h(z) + (-1)^n \overline{\mathcal{D}^n g(z)}, \tag{3}$$

where

$$\mathcal{D}^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad \mathcal{D}^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k,$$

(see also [14]).

Several authors such as ([4], [5], [6], [8], [11] and [17]) introduced and studied various new subclasses of analytic univalent as well as harmonic univalent functions with the help of convolution.

Motivated with the earlier introduced subclasses of $\mathcal{S}_H$, in this paper, we define a generalized class $\mathcal{S}_H^i(m, n, \phi, \psi; \alpha)$ of functions $f = h + \overline{g} \in \mathcal{S}_H$ satisfying for $i \in \{0, 1\}$, the condition

$$\Re\left\{\frac{D^m h(z) * \phi(z) + (-1)^{m+i}\overline{D^m g(z) * \psi(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}}\right\} > \alpha, \tag{4}$$

where $m, n \in \mathbb{N}_0$, $m \geq n$, $0 \leq \alpha < 1$, and $\phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ are analytic in $\mathbb{U}$ with the conditions $\lambda_k \geq 1$, $\mu_k \geq 1$. The operator "*" stands for the Hadamard product or convolution of two power series.

We further denote by $\mathcal{TS}_H^i(m, n, \phi, \psi; \alpha)$, a subclass of $\mathcal{S}_H^i(m, n, \phi, \psi; \alpha)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_H$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \ g(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{5}$$

It is interesting to note that by specializing the parameters we obtain the following known subclasses of $\mathcal{S}_H$ studied earlier by various researchers.

(i) $\mathcal{S}_H^0(m, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{S}_H(m, n; \alpha)$ and $\mathcal{TS}_H^0(m, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{TS}_H(m, n; \alpha)$ studied by Yalcin [17].

(ii) $\mathcal{S}_H^0(n+1, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{S}_H(n;\alpha)$ and $\mathcal{TS}_H^0(n+1, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{TS}_H(n;\alpha)$ studied by Jahangiri et al. [10].

(iii) $\mathcal{S}_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{S}_H^1(0, 0, \frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha) \equiv \mathcal{S}_H^*(\alpha)$ and $\mathcal{TS}_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{TS}_H^1(0, 0, \frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha) \equiv \mathcal{TS}_H^*(\alpha)$ studied by Jahangiri [9].

(iv) $\mathcal{S}_H^0(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{K}_H(\alpha)$ and $\mathcal{TS}_H^0(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) \equiv \mathcal{TK}_H(\alpha)$ studied by Jahangiri [9].

(v) $\mathcal{S}_H^1(0, 0, \phi, \psi; \alpha) \equiv \mathcal{S}_H(\phi, \psi; \alpha)$ and $\mathcal{TS}_H^1(0, 0, \phi, \psi; \alpha) \equiv \mathcal{TS}_H(\phi, \psi; \alpha)$ studied by Frasin [8].

(vi) $\mathcal{S}_H^0(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; 0) \equiv \mathcal{K}_H$, $\mathcal{TS}_H^0(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; 0) \equiv \mathcal{TK}_H$, $\mathcal{S}_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; 0) \equiv \mathcal{S}_H^*$ and $\mathcal{TS}_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; 0) \equiv \mathcal{TS}_H^*$ studied by Silverman [15], Silverman and Silvia [16](see also [2]).

In the present paper, we prove a number of sharp results including, coefficient inequality, bounds, extreme points, convolution and convex combination for functions in $\mathcal{TS}_H^i(m, n, \phi, \psi; \alpha)$ under certain conditions on the coefficients of ϕ and ψ.

2. Main Results

We begin with a sufficient coefficient condition for functions to be in class $\mathcal{S}_H^i(m, n, \phi, \psi; \alpha)$.

Theorem 1. *Let a function $f = h + \overline{g}$, where h and g are of the form (1), satisfies*

$$\sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k| \leq 1, \tag{6}$$

where $i \in \{0,1\}, m \in \mathbb{N}, n \in \mathbb{N}_0$, $m \geq n$, $\lambda_k, \mu_k \geq 1, k \geq 1, 0 \leq \alpha < 1$ and in case $m = 0 = n, \lambda_k, \mu_k \geq k$, $k \geq 1$. Then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f \in \mathcal{S}_H^i(m, n, \phi, \psi; \alpha)$.

Proof. Under the given hypothesis, we note that for $k \geq 1$,

$$k \leq \frac{\lambda_k k^m - \alpha k^n}{1-\alpha}, k \leq \frac{\mu_k k^m - (-1)^{m+i-n}\alpha k^n}{1-\alpha} . \tag{7}$$

Hence, for $f = h + \overline{g}$, where h and g are of the form (1), we get that

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| > 1 - \sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha}|a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k| \geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k|r^{k-1} \geq |g'(z)|, \end{aligned}$$

which proves that f is sense-preserving in $\mathbb{U}$. To show that f is univalent in $\mathbb{U}$, suppose $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$, then

$$\begin{aligned} \left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| \\ &= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right| \\ &> 1 - \left|\frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|}\right| \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha}|a_k|} \geq 0. \end{aligned}$$

Now, to show that $f \in \mathcal{S}_H^i(m, n, \phi, \psi; \alpha)$, we use the fact that $\mathrm{Re}\{\omega\} \geq \alpha$, if and only if $|1 - \alpha + \omega| \geq |1 + \alpha - \omega|$.

Hence, it suffices to show that

$$Q(z) := |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0, \tag{8}$$

where $A(z) = D^m h(z) * \phi(z) + (-1)^{m+i}\, \overline{D^m g(z) * \psi(z)}$ and $B(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}$.

Substituting the corresponding series expansions in the expressions of $A(z)$ and

$B(z)$, we obtain from (8), that

$$
\begin{aligned}
Q(z) &= \Bigg|(2-\alpha)z+\sum_{k=2}^{\infty}(k^m\lambda_k+(1-\alpha)k^n)a_kz^k+\\
&\quad (-1)^{m+i}\sum_{k=1}^{\infty}[k^m\mu_k+(-1)^{m+i-n}(1-\alpha)k^n]\overline{b_kz^k}\Bigg|\\
&\quad -\Bigg|-\alpha z+\sum_{k=2}^{\infty}[k^m\lambda_k-(1+\alpha)k^n]a_kz^k+\\
&\quad (-1)^{m+i}\sum_{k=1}^{\infty}[k^m\mu_k-(-1)^{m+i-n}(1+\alpha)k^n]\overline{b_kz^k}\Bigg|\\
&> 2|z|\left[(1-\alpha)-\sum_{k=2}^{\infty}(k^m\lambda_k-\alpha k^n)|a_k|-\sum_{k=1}^{\infty}[k^m\mu_k-(-1)^{m+i-n}\alpha k^n]|b_k|\right]\\
&\geq 0,
\end{aligned}
$$

if (6) holds. This proves the Theorem 1.

Sharpness of the coefficient inequality (6) can be seen by the function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda_kk^m-\alpha k^n}x_kz^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{\mu_kk^m-(-1)^{m+i-n}\alpha k^n}\overline{y_kz^k},$$

where $i\in\{0,1\},0\leq\alpha<1,\ m\in\mathbb{N},n\in\mathbb{N}_0,\ m\geq n,\lambda_k,\mu_k\geq 1,k\geq 1$ in case $m=0=n,\lambda_k,\mu_k\geq k,\ k\geq 1$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$.

We next show that the above sufficient coefficient condition is also necessary for functions in the class $\mathcal{TS}_H^i(m,n,\phi,\psi;\alpha)$.

Theorem 2. *Let the function $f=h+\overline{g}$ be such that h and g are given by (5). Then, $f\in\mathcal{TS}_H^i(m,n,\phi,\psi;\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{\lambda_kk^m-\alpha k^n}{1-\alpha}|a_k|+\frac{\mu_kk^m-(-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k|\right)\leq 2, \tag{9}$$

where $a_1=1,\ m\in\mathbb{N},n\in\mathbb{N}_0,\ m\geq n,\ \lambda_k,\mu_k\geq 1,k\geq 1,0\leq\alpha<1$ and in case $m=0=n,\ \lambda_k,\mu_k\geq k,\ k\geq 1$.

Proof. The if part, follows from Theorem 1. To prove the "only if" part, let $f\in\mathcal{TS}_H^i(m,n,\phi,\psi;\alpha)$, then from (4), we have

$$\Re\left\{\frac{D^mh(z)*\phi(z)+(-1)^{m+i}\overline{D^mg(z)*\psi(z)}}{D^nh(z)+(-1)^n\overline{D^ng(z)}}-\alpha\right\}>0, z\in\mathbb{U},$$

which is equivalent to

$$\Re\left\{\frac{\begin{array}{c}(1-\alpha)z-\sum_{k=2}^{\infty}(\lambda_k k^m-\alpha k^n)|a_k|z^k+\\(-1)^{2m+2i-1}\sum_{k=1}^{\infty}(\mu_k k^m-(-1)^{m+i-n}\alpha k^n)|b_k|\overline{z}^k\end{array}}{z-\sum_{k=2}^{\infty}k^n|a_k|z^k+(-1)^{m+i-1+n}\sum_{k=1}^{\infty}k^n|b_k|\overline{z}^k}\right\}>0.$$

If we choose z to be real and $z\to 1^-$, we get

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}(\lambda_k k^m-\alpha k^n)|a_k|-\sum_{k=1}^{\infty}(\mu_k k^m-(-1)^{m+i-n}\alpha k^n)|b_k|}{1-\sum_{k=2}^{\infty}k^n|a_k|+(-1)^{m+i-1+n}\sum_{k=1}^{\infty}k^n|b_k|}\geq 0$$

or, equivalently,

$$\sum_{k=2}^{\infty}(\lambda_k k^m-\alpha k^n)|a_k|+\sum_{k=1}^{\infty}(\mu_k k^m-(-1)^{m+i-n}\alpha k^n)|b_k|\leq 1-\alpha,$$

which is the required condition (9).

For the classes $\mathcal{TS}_H(m,n;\alpha)$ and $\mathcal{TS}_H(\phi,\psi;\alpha)$ mentioned in Section 1, Theorem 2 yields following results which include the results for other known classes discussed in Section 1.

Corollary 3. *[17] Let the function $f=h+\overline{g}$ be such that h and g are given by (5). Then, $f\in\mathcal{TS}_H(m,n;\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{k^m-\alpha k^n}{1-\alpha}|a_k|+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|\right)\leq 2, \tag{10}$$

where $a_1=1,\ m\in\mathbb{N},n\in\mathbb{N}_0,\ m\geq n,0\leq\alpha<1$.

Corollary 4. *Let the function $f=h+\overline{g}$ be such that h and g are given by (5). Then, $f\in\mathcal{TS}_H(\phi,\psi;\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{\lambda_k-\alpha}{1-\alpha}|a_k|+\frac{\mu_k+\alpha}{1-\alpha}|b_k|\right)\leq 2, \tag{11}$$

where $a_1=1,\ \lambda_k,\mu_k\geq k,\ k\geq 1,0\leq\alpha<1$.

3. Bounds

Our next theorem provides the bounds for the functions in $\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$ which is followed by a covering result for this class.

Theorem 5. *Let $f = h+\overline{g}$ with h and g are of the form (5) belongs to the class $\mathcal{TS}^i_H(m,n,\phi,\psi;\alpha)$ for functions ϕ and ψ with non-decreasing sequences $\{\lambda_k\},\{\mu_k\}$ satisfying $\lambda_k,\mu_k \geq \lambda_2$, $k\geq 2$, then*

$$|f(z)| \leq (1+|b_1|)r+\left(1-\frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{2^m\lambda_2-\alpha 2^n}, \qquad |z|=r<1, \tag{12}$$

and

$$|f(z)| \geq (1-|b_1|)r-\left(1-\frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{2^m\lambda_2-\alpha 2^n}, \qquad |z|=r<1. \tag{13}$$

Proof. We only prove the result for upper bound. The result for the lower bound can similarly be obtained.

Let $f\in \mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$, then on taking the absolute value of f, we get for $|z|=r<1$,

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\
&\leq (1+|b_1|)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\
&\leq (1+|b_1|)r+\frac{(1-\alpha)\,r^2}{2^m\lambda_2-\alpha 2^n}\sum_{k=2}^{\infty}\left(\frac{\lambda_k k^m-\alpha k^n}{1-\alpha}|a_k|+\frac{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k|\right)\\
&\leq (1+|b_1|)r+\left(1-\frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{2^m\lambda_2-\alpha 2^n}, \quad \text{by (9)}.
\end{aligned}$$

The bounds (12) and (13) are sharp for the function given by

$$f(z)=z+|b_1|\overline{z}+\left(1-\frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,\overline{z}^2}{2^m\lambda_2-\alpha 2^n} \tag{14}$$

for $|b_1|<(1-\alpha)/(1-(-1)^{m+i-n}\alpha)$.

A covering result follows from (13).

Corollary 6. *Let* $f \in \mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$*, then for functions* ϕ *and* ψ *with non-decreasing sequences* $\{\lambda_k\},\{\mu_k\}$ *satisfying* $\lambda_k,\mu_k \geq \lambda_2,\ k \geq 2$*,*

$$\left\{\omega : |\omega| < \left(1 - \frac{(1-\alpha)}{2^m\lambda_2 - \alpha 2^n}\right) + \left(\frac{1-(-1)^{m+i-n}\alpha}{2^m\lambda_2 - \alpha 2^n} - 1\right)|b_1|\right\} \subset f(\mathbb{U}).$$

Further, for the classes $\mathcal{TS}_H(m,n;\alpha)$ and $\mathcal{TS}_H(\phi,\psi;\alpha)$,Theorem 5 yields following results which include the results for other known classes discussed in Section 1.

Corollary 7. *[17] Let* $f = h + \overline{g}$ *with* h *and* g *are of the form (5) belongs to the class* $\mathcal{TS}_H(m,n;\alpha)$*, then*

$$|f(z)| \leq (1+|b_1|)r + \left(1 - \frac{1-(-1)^{m-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{2^m - \alpha 2^n}, \qquad |z| = r < 1, \quad (15)$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(1 - \frac{1-(-1)^{m-n}\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{2^m - \alpha 2^n}, \qquad |z| = r < 1. \quad (16)$$

Further,

$$\left\{\omega : |\omega| < \left(1 - \frac{1-\alpha}{2^m - \alpha 2^n}\right) + \left(\frac{1-(-1)^{m-n}\alpha}{2^m - \alpha 2^n} - 1\right)|b_1|\right\} \subset f(\mathbb{U}).$$

Corollary 8. *Let* $f = h + \overline{g}$ *with* h *and* g *are of the form (5) belongs to the class* $\mathcal{TS}_H(\phi,\psi;\alpha)$ *for functions* ϕ *and* ψ *with non-decreasing sequences* $\{\lambda_k\},\{\mu_k\}$ *satisfying* $\lambda_k,\mu_k \geq \lambda_2,\ k \geq 2$*, then*

$$|f(z)| \leq (1+|b_1|)r + \left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{\lambda_2 - \alpha}, \qquad |z| = r < 1, \qquad (17)$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)\frac{(1-\alpha)\,r^2}{\lambda_2 - \alpha}, \qquad |z| = r < 1. \qquad (18)$$

Further,

$$\left\{\omega : |\omega| < \frac{1}{\lambda_2 - \alpha}\left(\lambda_2 - 1 + (1 - \lambda_2 + 2\alpha)\,|b_1|\right)\right\} \subset f(\mathbb{U}).$$

4. Extreme Points

In this section we determine the extreme points of $\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$.

Theorem 9. *Let* $h_1(z)=z,\ h_k(z)=z-\frac{1-\alpha}{\lambda_k k^m-\alpha k^n}z^k$ $(k\geq 2)$ *and* $g_k(z)=z+\frac{(-1)^{m+i-1}(1-\alpha)}{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}\overline{z}^k$ $(k\geq 1)$*. Then* $f\in\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$*, if and only if it can be expressed as*

$$f(z)=\sum_{k=1}^{\infty}(x_k h_k(z)+y_k g_k(z)), \tag{19}$$

where $x_k\geq 0, y_k\geq 0$ *and* $\sum_{k=1}^{\infty}(x_k+y_k)=1$*. In particular, the extreme points of* $\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$ *are* $\{h_k\}$ *and* $\{g_k\}$*.*

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}(x_k h_k(z)+y_k g_k(z)).$$

Then,

$$\begin{aligned}f(z)=&\sum_{k=1}^{\infty}(x_k+y_k)z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda_k k^m-\alpha k^n}x_k z^k+\\ &(-1)^{m+i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}y_k\overline{z}^k\\ &=z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda_k k^m-\alpha k^n}x_k z^k+(-1)^{m+i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}y_k\overline{z}^k\\ &\in\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha).\end{aligned}$$

Since,

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\lambda_k k^m-\alpha k^n}{1-\alpha}\frac{1-\alpha}{\lambda_k k^m-\alpha k^n}x_k\\ &+\sum_{k=1}^{\infty}\frac{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}{1-\alpha}\frac{1-\alpha}{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}y_k\\ &=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\\ &=1-x_1\leq 1.\end{aligned}$$

Conversely, if $f\in\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$, then $|a_k|\leq\frac{1-\alpha}{\lambda_k k^m-\alpha k^n}$, $k\geq 2$ and $|b_k|\leq\frac{1-\alpha}{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}$, $k\geq 1$. Setting $x_k=\frac{\lambda_k k^m-\alpha k^n}{1-\alpha}|a_k|$, $k\geq 2$ and $y_k=$

$\frac{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}{1-\alpha}|b_k|,\ k \geq 1$.Then, by Theorem 2, $\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \geq 0$. Consequently, we can see that $f(z)$ can be expressed in the form (19).

This completes the proof of Theorem 9.

5. Convolution and Convex Combinations

In this section, we show that the class $\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$ is invariant under convolution and convex combinations of its members.

For harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |B_k| \overline{z}^k,$$

we define the convolution

$$(f*F)(z) = f(z)*F(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k.$$

Theorem 10. *If $f \in \mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$ and $F \in \mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$ then $f*F \in \mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$.*

Proof. Let $f(z) = z-\sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |B_k| z^k$ be in $\mathcal{TS}^i{}_H(m,n,\phi,\psi;\alpha)$. Then by Theorem 2, we have

$$\sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |b_k| \leq 1, \tag{20}$$

and

$$\sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |A_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |B_k| \leq 1. \tag{21}$$

From (21), we conclude that $|A_k| \leq 1,\ k = 2,3,...$ and $|B_k| \leq 1,\ k = 1,2,...$

So, for $f*F$, we may write

$$\sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |a_k A_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |b_k B_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |b_k| \leq 1.$$

Thus $f * F \in \mathcal{TS}^i{}_H(m, n, \phi, \psi; \alpha)$.

Finally, we prove that $\mathcal{TS}^i{}_H(m, n, \phi, \psi; \alpha)$ is closed under convex combination of its members.

Theorem 11. *The class $\mathcal{TS}^i{}_H(m, n, \phi, \psi; \alpha)$ is closed under convex combination.*

Proof. For $j = 1, 2, ...$ suppose that $f_j \in \mathcal{TS}^i{}_H(m, n, \phi, \psi; \alpha)$ where $f_j(z)$ is given by

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{j,k}| z^k + (-1)^{m+i-n} \sum_{k=1}^{\infty} |b_{j,k}| z^k.$$

Then, by Theorem 2, we have

$$\sum_{k=1}^{\infty} \left(\frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |a_{j,k}| + \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |b_{j,k}| \right) \leq 2. \tag{22}$$

For $\sum_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the convex combination of $f_j(z)$ may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} t_j |a_{j,k}| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} t_j |b_{j,k}| z^k.$$

Now

$$\begin{aligned} &\sum_{k=1}^{\infty} \left(\frac{\lambda_k k^m - \alpha k^n}{1-\alpha} \sum_{j=1}^{\infty} t_j |a_{j,k}| + \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} \sum_{j=1}^{\infty} t_j |b_{j,k}| \right) \\ &= \sum_{j=1}^{\infty} t_i \sum_{k=1}^{\infty} \left(\frac{\lambda_k k^m - \alpha k^n}{1-\alpha} |a_{j,k}| + \frac{\mu_k k^m - (-1)^{m+i-n} \alpha k^n}{1-\alpha} |b_{j,k}| \right) \\ &\leq 2 \sum_{j=1}^{\infty} t_i = 2 \end{aligned}$$

and so by Theorem 2, we have $\sum_{j=1}^{\infty} t_i f_i(z) \in \mathcal{TS}^i{}_H(m, n, \phi, \psi; \alpha)$.

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