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ON PRESERVING ΩS-CLOSENESS IN TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to introduce and study the concepts of Ωs^*-closed and Ωs^*-continuous maps. These concepts are used to obtain several results concerning the preservation of Ωs-closed sets. Moreover, we use Ωs^*-closed and Ωs^*-continuous maps to obtain a characterization of $\Omega - T_{\frac{1}{2}}$ spaces.

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1. Introduction

Noiri and Sayed [16] introduced the class of Ωs-closed sets. By the mean of these sets they introduced and studied Ωs-continuous and Ωs-irresolute maps. In [17], Sayed introduced Ωs-open sets and studied some applications on them. In this paper, we obtain some new decompositions of Ωs-continuity. Also, a new forms of continuity (which we call Ωs^*-closed and Ωs^*-continuous) are introduced and several properties of them are investigated. We use these concepts to obtain some results concerning the preservation of Ωs-closed sets. Furthermore, we characterize $\Omega - T_{\frac{1}{2}}$ and $semi - T_{\frac{1}{2}}$ spaces in terms of Ωs-closed sets.

2. Preliminaries

Throughout this paper (X, τ) , (Y, σ) and (Z, υ) represent non-empty spaces on which no separation axioms are assumed, unless otherwise mentioned, and they are simply written as X , Y, and Z, respectively, when no confusion arises.The family of all closed subsets of X (resp. Y) is denoted by F_X (resp. F_Y). All sets are assumed to be subsets of topological spaces. The closure and the interior of a set A are denoted by $Cl(A)$ [6] and $Int(A)$ [7], respectively. In order to make the contents of this paper as self contained as possible, we briefly describe certain definitions; notations and some properties.

Definition 1. *A subset A of (X, τ) is said to be:*

(1) semi-open [11] *if $A \subseteq Cl(Int(A))$ and semi-closed* [4] *if $Int(Cl(A)) \subseteq A$;*

(2) preopen [14] *if $A \subseteq Int(Cl(A))$;*

(3) semi-preopen [1] *if $A \subseteq Cl(Int(Cl(A)))$;*

(4) regular open (resp. regular closed) [19] *if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$).*

Definition 2. *Let (X, τ) be a topological space and $A \subseteq X$. The semi-interior of A* [6], *denoted by $sInt(A)$, is the union of all semi-open subsets of A. A is semi-open* [6] *if and only if $sInt(A) = A$. It is well Known that $sInt(A) = A \cap Cl(Int(A))$* [10].

Definition 3. *Let (X, τ) be a topological space and $A, B \subseteq X$. Then A is semi-closed if and only if $X \backslash A$ is semi-open and the semi-closure of B* [4], *denoted by $sCl(B)$, is the intersection of all semi-closed supersets of B. B is semi-closed* [15] *if and only if $sCl(B) = B$. It is well Known that $sCl(B) = B \cup Int(Cl(B))$* [10].

Definition 4. *A subset A of (X, τ) is said to be*

(1) sg-closed [3] *in (X, τ) if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ);*

(2) Ω-closed [16] *in (X, τ) if $sCl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is semi-open in (X, τ);*

(3) Ωs-closed [16] *in (X, τ) if $sCl(A) \subseteq Int(Cl(U)))$ whenever $A \subseteq U$ and U is semi-open in (X, τ).*

(4) The complement of Ω-closed set (resp. Ωs-closed set) is said to be Ω-open (resp. Ωs-open) [17] *in (X, τ). Equivalently, a subset A of a space (X, τ) is said to be Ωs-open* [17, *Proposition 2.3(2)*] *if $Cl(Int(F)) \subseteq sInt(A)$ whenever $F \subseteq A$ and F is semi-closed.*

We need the following notations:

• $\Omega_s C(X, \tau)$ (resp. $\Omega_s O(X, \tau)$) denotes the family of all Ωs-closed sets (resp. Ωs-open sets) in (X, τ);

• $\Omega C(X, \tau)$ (resp. $\Omega O(X, \tau)$) denotes the family of all Ω-closed sets (resp. Ω-open sets) in (X, τ);

• $SGC(X, \tau)$ (resp. $SGO(X, \tau)$) denotes the family of all *sg*-closed sets (resp. *sg*-open sets) in (X, τ);

• $SC(X, \tau)$ (resp. $SO(X, \tau)$) denotes the family of all semi-closed sets (resp. semi-open sets) in (X, τ);

Definition 5. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

(1) RC-continuous [2] *(resp. contra-semicontinuous* [8], *contra-precontinuous* [9]*) if $f^{-1}(V)$ is regular closed (resp. semi-closed, preclosed) in (X,τ) for every open subset V in (Y,σ);*

(2) Ωs-continuous [16] *(resp. sg-continuous* [20]*) if $f^{-1}(V)$ is Ωs-closed (resp. sg-closed) in (X,τ) for every closed subset V in (Y,σ);*

(3) irresolute [7]*(resp. Ωs-irresolute* [16]*) if $f^{-1}(V)$ is semi-open (resp. Ωs-closed) in (X,τ) for every semi-open (resp. Ωs-closed) subset V in (Y,σ);*

(4) pre-semiopen [7]*(resp. pre-semi-closed* [18], *pre-Ωs-closed* [17]*) if $f(F)$ is semi-open (resp. semi-closed, Ωs-closed) in (Y,σ) whenever F is semi-open (resp. semi-closed, Ωs-closed) in (X,τ).*

Definition 6. *A topological space (X,τ) is said to be semi $-T_{\frac{1}{2}}$* [5] *(resp. $\Omega - T_{\frac{1}{2}}$* [15]*) if every sg-closed (resp. Ωs-closed) set is semi-closed.*

3. Ωs-closed sets and Ωs-continuity

Theorem 1. *Every preopen subset of (X,τ) is Ωs-closed.*

Proof. Let A be a preopen subset of (X,τ) and $A \subseteq U$, where U is a semi-open set in (X,τ). Then $sCl(A) = A \cup Int(Cl(A)) = Int(Cl(A)) \subseteq Int(Cl(U))$. Hence A is Ωs-closed.

Remark 1. *We have the following more relationship between Ωs-closed sets and some other sets (cf. Remark 3.2 in [16]); and the following examples below show them.*

1) An Ωs-closed set need not be pre-open (cf. Example 1);

(2) Semi-preopen sets and Ωs-closed sets are independent (cf. Examples 1 and 2);

(3) Semi-closed sets and Ωs-closed sets are independent (cf. Examples 1 and 2);

(4) A closed semi-open set need not be Ωs-closed (cf. Example 2);

(5) sg-closed sets and Ωs-closed sets are independent (cf. Example 2).

Example 1. *Let $X = \{a,b\}$ be the Sierpinski space and $\tau = \{X,\phi,\{a\}\}$. The subset $\{b\}$ of X is Ωs-closed but it is neither preopen nor semi-preopen. Furthermore, the subset $\{a\}$ of X is Ωs-closed but it is not semi-closed.*

Example 2. *Let $X = \{a,b,c\}$ and $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. The subset $\{b,c\}$ of X is both closed and semi-open but it is not Ωs-closed. Also, $\{a,b\}$ is Ωs-closed but it is not sg-closed. Furthermore, the subset $\{c\}$ of X is sg-closed but it is not Ωs-closed.*

Theorem 2. *If a subset A of a space (X, τ) is regular open then A is both semi-open and Ωs-closed and the converse is not true.*

Proof. Let A be a regular open subset of (X, τ). Then A is semi-open in (X, τ). To prove that A is Ωs-closed, let $A \subseteq G$, where G is a semi-open subset of (X, τ). Then $sCl(A) = A \cup Int(Cl(A)) = Int(Cl(A)) \subseteq Int(Cl(G))$. Therefore A is Ωs-closed.
Conversely, in Example 2 the subset $\{a, b\}$ of X is both semi-open and Ωs-closed but it is not regular open.

Corollary 3. *If a subset A of a space (X, τ) is regular closed then it is both semi-closed and Ωs-open.*

Remark 2. *The converse of the above corollary is not true as shown by Example 2, where $\{c\}$ is both semi-closed and Ωs-open but it is not regular closed .*

Corollary 4. *A subset A of a space (X, τ) is clopen if and only if A is semi-open, semi-closed, Ωs-open and Ωs-closed.*

Theorem 5. *A contra-precontinuous map is Ωs-continuous.*

Proof. From Theorem 1, the proof is straightforward.

The converse of the above theorem is not true as shown by the following example

Example 3. *Let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$. The identity map $f : (X, \tau) \longrightarrow (X, \tau)$ is Ωs-continuous but it is not contra-precontinuous.*

Theorem 6. *If the map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is RC-continuous, then it is both Ωs-continuous and contra-semicontinuous.*

Proof. From Theorem 2, the proof is straightforward.

The converse of the above theorem is not true as shown by the following example

Example 4. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \phi, \{c\}\}$. Define $f : (X, \tau) \longrightarrow (X, \sigma)$ to be the identity map. Then f is both contra-semicontinuous and $\Omega s-$continuous but not RC-continuous.*

Let (X, τ) be a topological space. If $\tau = F_X$, then
(1) $SO(X, \tau) = SC(X, \tau) = \tau$.
(2) $\Omega_s O(X, \tau) = \Omega_s C(X, \tau) = P(X)$.

4. Ωs^*-closed and Ωs^*-continuous maps

In this section, we introduce a new type of maps called Ωs^*-closed and Ωs^*-continuous maps and obtain some of their properties and characterizations. Furthermore, we establish a necessary and sufficient conditions for a map to be Ωs^*-closed and Ωs^*-continuous.

Definition 7. *A map $f:(X,\tau)\to(Y,\sigma)$ is said to be Ωs^*-closed if $f(Cl(Int(S)))\subseteq sInt(O)$, whenever S is a semi-closed subset of (X,τ), O is an Ωs-open subset of (Y,σ) and $f(S)\subseteq O$.*

Definition 8. *A map $f:(X,\tau)\to(Y,\sigma)$ is said to be Ωs^*-continuous if $sCl(O_1)\subseteq f^{-1}(Int(Cl(S_1)))$, whenever O_1 is an Ωs-closed subset of (X,τ), S_1 is a semi-open subset of (Y,σ) and $O_1\subseteq f^{-1}(S_1)$.*

The following example shows that Ωs^*-continuous is not continuous, not Ωs^*-closed, and not Ωs-irresolute.

Example 5. *Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ be a topology on X , and $\sigma=\{Y,\phi,\{a\},\{b,c\}\}$ be a topology on Y. Define the map $f:(X,\tau)\to(Y,\sigma)$ to be the identity map. We have that f is Ωs^*-continuous but not continuous, not Ωs^*-closed, not Ωs-irresolute, and not Ωs-continuous.*

The following example shows that Ωs-continuous does not imply Ωs^*-continuous.

Example 6. *Let $X=Y=\{a,b,c\}$, $\tau=\{X,\phi,\{c\}\}$ be a topology on X , and $\sigma=\{Y,\phi,\{a\},\{b,c\}\}$ be a topology on Y. Define the map $f:(X,\tau)\to(Y,\sigma)$ to be the identity map.We have that f is Ωs-continuous, but not Ωs^*-continuous.*

From the above discussion we note that:
(1) Ωs-continuity and Ωs^*-continuity are independent.
(2) Continuity and Ωs^*-continuity are independent.

Theorem 7. *For a map $f:(X,\tau)\to(Y,\sigma)$, we denote the following properties by (1), (2) and (3), respectively.*

(1) $f:(X,\tau)\to(Y,\sigma)$ is Ωs^*-closed;

(2) $sCl(O_1)\subseteq f(Int(Cl(S_1)))$ holds, whenever S_1 is a semi-open subset of (X,τ), O_1 is an Ωs-closed subset of (Y,σ) and $O_1\subseteq f(S_1)$;

(3) $Cl(Int(S))\subseteq f^{-1}(sInt(O))$ holds, whenever S is a semi-closed subset of (X,τ), O is an Ωs-open subset of (Y,σ) and $S\subseteq f^{-1}(O)$.

Then, we have the following implications:

(i) (2)$\Rightarrow$(1) if $f:(X,\tau)\to(Y,\sigma)$ is surjective;
(ii) (1)$\Rightarrow$(2) if $f:(X,\tau)\to(Y,\sigma)$ is bijective;
(iii) (1)$\Leftrightarrow$(3).

Proof. (i) Let $S \in SC(X,\tau)$ and $O \in \Omega_s O(Y,\sigma)$ such that $f(S) \subseteq O$. Then, since f is surjective, we have that $Y \backslash O \subseteq Y \backslash f(S) \subseteq f(X \backslash S)$. For the sets $X \backslash S \in SO(X,\tau)$ and $Y \backslash O \in \Omega_s C(Y,\sigma)$, by (2), it is obtained that $sCl(Y \backslash O) \subseteq f(Int(Cl(X \backslash S)))$; and so $f(Cl(Int(S))) \subseteq sInt(O)$. Therefore, $f:(X,\tau) \to (Y,\sigma)$ is Ωs^*-closed.

(ii) Let $S_1 \in SO(X,\tau)$ and $O_1 \in \Omega_s C(Y,\sigma)$ such that $O_1 \subseteq f(S_1)$. Then, since f is injective, we have that $f(X \backslash S_1) \subseteq Y \backslash f(S_1) \subseteq Y \backslash O_1$. For the sets $X \backslash S_1 \in SC(X,\tau)$ and $Y \backslash O_1 \in \Omega_s O(Y,\sigma)$, by (1), it is obtained that $f(Cl(Int(X \backslash S_1))) \subseteq sInt(Y \backslash O_1)$; and so $f(X \backslash Int(Cl(S_1))) \subseteq Y \backslash sCl(O_1)$. Using the assumption of surjectivity of f, we have that $Y \backslash f(Int(Cl(S_1))) \subseteq f(X \backslash Int(Cl(S_1))) \subseteq Y \backslash sCl(O_1)$ and so $sCl(O_1) \subseteq f(Int(Cl(S_1)))$.

(iii) (1)⇒(3) Let $S \in SC(X,\tau)$ and $O \in \Omega_s O(Y,\sigma)$ such that $S \subseteq f^{-1}(O)$. Since f is Ωs^*-closed, we have $f(Cl(Int(S))) \subseteq sInt(O)$; and so $Cl(Int(S)) \subseteq f^{-1}(sInt(O))$.

(3)⇒(1) Let $S \in SC(X,\tau)$ and $O \in \Omega_s O(Y,\sigma)$ such that $f(S) \subseteq O$. Since $S \subseteq f^{-1}(O)$, by (3), it is obtained that $Cl(Int(S)) \subseteq f^{-1}(sInt(O))$ holds; and so $f(Cl(Int(S))) \subseteq sInt(O)$.

Theorem 8. *For a map $f:(X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:*

(1) f is Ωs^*-continuous.

(2) $f^{-1}(Cl(Int(S))) \subseteq sInt(O)$ whenever $f^{-1}(S) \subseteq O$, where S is a semi-closed subset of Y and O is an Ωs-open subset of X.

(3) $f(sCl(O_1)) \subseteq Int(Cl(S_1))$ whenever $f(O_1) \subseteq S_1$, where O_1 is an Ωs-closed subset of X and S_1 is a semi-open subset of Y.

Proof. (1)⟹(2) Suppose that $f^{-1}(S) \subseteq O$, where $S \in SC(Y,\sigma)$ and $O \in \Omega_s O(X,\tau)$. Since $X \backslash O \subseteq f^{-1}(Y \backslash S)$ and f is Ωs^*-continuous, then $X \backslash sInt(O) = sCl(X \backslash O) \subseteq f^{-1}(Int(Cl(Y \backslash S))) = X \backslash f^{-1}(Int(Cl(S)))$. Therefore, we have the required property: $f^{-1}(Int(Cl(S))) \subseteq sInt(O)$.

(2)⟹(3) Let $f(O_1) \subseteq S_1$, where $S_1 \in SO(Y,\sigma)$ and $O_1 \in \Omega_s C(X,\tau)$. Then, we have $f^{-1}(Y \backslash S_1) \subseteq X \backslash O_1, Y \backslash S_1 \in SC(Y,\sigma)$ and $X \backslash O_1 \in \Omega_s O(X,\tau)$. By (2), it is obtained that $X \backslash f^{-1}(Int(Cl(S_1))) = f^{-1}(Cl(Int(Y \backslash S_1))) \subseteq sInt(X \backslash O_1) = X \backslash sCl(O_1)$; and so $f(sCl(O_1)) \subseteq Int(Cl(S_1))$.

(3)⇒(1) Let $S \in SO(Y,\sigma)$ and $O \in \Omega_s C(X,\tau)$ such that $O \subseteq f^{-1}(S)$. Since $f(O) \subseteq f(f^{-1}(S)) \subseteq S$, by (3), it is obtained that $f(sCl(O)) \subseteq Int(Cl(S))$ and hence $sCl(O) \subseteq f^{-1}(Int(Cl(S)))$. Therefore, f is Ωs^*-continuous.

Theorem 9. *Let $f:(X,\tau) \to (Y,\sigma)$ be a bijection. Then the following conditions are equivalent:*

(1) f is Ωs^*-closed.

(2) f^{-1} is Ωs^*-continuous.

Proof. (1)$\Longrightarrow$(2) Let $O_1 \subseteq (f^{-1})^{-1}(S_1) = f(S_1)$, where O_1 is an Ωs-closed subset of (Y,σ) and S_1 is a semi-open subset of (X,τ). From Theorem 7 we have $sCl(O_1) \subseteq f(Int(Cl(S_1))) = (f^{-1})^{-1}(Int(Cl(S_1)))$. Hence f^{-1} is Ωs^*-continuous.

(2)$\Longrightarrow$(1) Let $O_1 \subseteq f(S_1)$ or $O_1 \subseteq (f^{-1})^{-1}(S_1)$, where O_1 is an Ωs-closed subset of (Y,σ) and S_1 is a semi-open subset of (X,τ). Then $sCl(O_1) \subseteq (f^{-1})^{-1}(Int(Cl(S_1)))$ or $sCl(O_1) \subseteq f(Int(Cl(S_1)))$. Therefore by Theorem 7 we have f is Ωs^*-closed.

Theorem 10. *Let $f : (X,\tau) \to (Y,\sigma)$ be a map . If $f(H)$ is a semi-closed subset of (Y,σ) and $f(Cl(Int(H))) \subseteq Cl(Int(f(H)))$ for every semi-closed subset H of (X,τ), then f is Ωs^*-closed map.*

Proof. Suppose that $f(H) \subseteq O$, where H is a semi-closed subset of (X,τ) and O is an Ωs-open subset of (Y,σ). Since O is an Ωs-open, then $Cl(Int(f(H))) \subseteq sInt(O)$. Hence $f(Cl(Int(H))) \subseteq sInt(O)$. Therefore f is Ωs^*-closed.

Theorem 11. *Let $f : (X,\tau) \to (Y,\sigma)$ be a map. If $f^{-1}(V)$ is a semi-open subset of (X,τ) and $Int(Cl(f^{-1}(V))) \subseteq f^{-1}(Int(Cl(V)))$ for every semi-open subset V of (Y,σ), then f is Ωs^*-continuous.*

Proof. Suppose that $O \subseteq f^{-1}(V)$, where O is an Ωs-closed subset of (X,τ) and V is a semi-open subset of (Y,σ). Since O is an Ωs-closed, then $sCl(O) \subseteq Int(Cl(f^{-1}(V)))$. Hence $sCl(O) \subseteq f^{-1}(Int(Cl(V)))$. Therefore f is Ωs^*-continuous.

5. Preserving Ωs-closed sets

In this section, the concepts of Ωs^*-closed and Ωs^*-continuous maps are used to study the preservation of Ωs-closed set. Also, we establish a necessary conditions for a map to be Ωs^*-closed and Ωs^*-continuous. Finally, we investigate some of the properties of these maps involving restriction and composition.

Theorem 12. *If $f : (X,\tau) \to (Y,\sigma)$ is irresolute and Ωs^*-closed, then $f^{-1}(B)$ is an Ωs-closed (Ωs-open) subset of (X,τ) whenever B is an Ωs-closed (Ωs-open) subset of (Y,σ).*

Proof. Assume that B is an Ωs-closed subset of (Y,σ) and $f^{-1}(B) \subseteq U$, where U is a semi-open subset of (X,τ). Then $X\backslash U \subseteq X\backslash f^{-1}(B) = f^{-1}(Y\backslash B)$ or $f(X\backslash U) \subseteq Y\backslash B$. Since f is Ωs^*-closed, then $f(Cl(Int(X\backslash U))) \subseteq sInt(Y\backslash B) = Y\backslash sCl(B)$. Hence $Cl(Int(X\backslash U)) \subseteq f^{-1}(Y\backslash sCl(B)) = X\backslash f^{-1}(sCl(B))$. Thus $f^{-1}(sCl(B)) \subseteq X\backslash Cl(Int((X\backslash U)) = Int(Cl(U))$. Since f is irresolute, then $sCl(f^{-1}(B)) \subseteq sCl(f^{-1}(sCl(B))) = f^{-1}(sCl(B)) \subseteq Int(Cl(U))$. Therefore $f^{-1}(B)$ is an Ωs-closed subset of (X,τ).

A similar argument shows that the inverse image of an Ωs-open set is an Ωs-open.

Remark 3. *From the above theorem we note that if* $f : (X, \tau) \to (Y, \sigma)$ *is irresolute and* Ωs^**-closed, then* f *is* $\Omega s-$*irresolute.*

The converse of the above remark is not true as illustrated by the following example

Example 7. *Let* $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}\}, Y = \{p, q\}$ *and* $\sigma = \{Y, \phi, \{p\}\}$. *Define* $f : (X, \tau) \to (Y, \sigma)$ *as follows:* $f(a) = f(c) = p$ *and* $f(b) = q$. *Then* f *is* $\Omega s-$*irresolute but not irresolute.*

Theorem 13. *If* $f : (X, \tau) \to (Y, \sigma)$ *is* Ωs^**-continuous and pre-semi-closed, then* $f(A)$ *is an* Ωs*-closed subset of* (Y, σ) *whenever* A *is an* Ωs*-closed subset of* (X, τ).

Proof. Assume that A is an Ωs-closed subset of (X, τ) and $f(A) \subseteq V$,where V is a semi-open subset of (Y, σ). Then $A \subseteq f^{-1}(V)$. Since f is Ωs^*-continuous, then $sCl(A) \subseteq f^{-1}(Int(Cl(V)))$. Hence $f(sCl(A)) \subseteq Int(Cl(V))$. Since f is pre-semi-closed, $sCl(f(A)) \subseteq sCl(f(sCl(A))) = f(sCl(A)) \subseteq Int(Cl(V))$. Therefore $f(A)$ is an Ωs-closed subset of (Y, σ) .

Remark 4. *From the above theorem we note that if* $f : (X, \tau) \to (Y, \sigma)$ *is* Ωs^**-continuous and pre-semi-closed, then* f *is pre-*Ωs*-closed.*

The converse of the above remark is not true as illustrated by the following example.

Example 8. *Let* $X = \{x, y\}, \tau = \{X, \phi, \{x\}\}, Y = \{p, q, r\}$ *and* $\sigma = \{Y, \phi, \{p\}, \{q, r\}\}$. *Define* $f : (X, \tau) \to (Y, \sigma)$ *as follows:* $f(x) = p$ *and* $f(y) = r$. *Then* f *is pre-*$\Omega s-$*closed but not pre-semi-closed.*

Theorem 14. *Let* (X, τ) *and* (Y, σ) *be two topological spaces such that* $\sigma = F_Y$. *If* $f : (X, \tau) \to (Y, \sigma)$ *is a pre-semi-closed map and* $f(Cl(Int(S))) \subseteq f(S)$ *holds for every semi-closed subset* S *of* (X, τ), *then* f *is an* Ωs^**-closed map.*

Proof. Let $f(S) \subseteq O$, where S be a semi-closed subset of (X, τ) and O is an Ωs-open subset of (Y, σ). Then, by Proposition 1, $f(S)$ is a semi-open subset of (Y, σ). Therefore $f(Cl(Int(S))) \subseteq f(S) \subseteq sInt(f(S)) \subseteq sInt(O)$. Hence $f : (X, \tau) \to (Y, \sigma)$ is an Ωs^*-closed map.

Theorem 15. *Let* (X, τ) *and* (Y, σ) *be two topological spaces such that* $\tau = F_X$. *If* $f : (X, \tau) \to (Y, \sigma)$ *is irresolute map and* $f^{-1}(S) \subseteq f^{-1}(Int(Cl(VS)))$ *holds for every semi-open subset* S *of* (Y, σ), *then* f *is* Ωs^**-continuous map.*

Proof. Let $O \subseteq f^{-1}(S)$, where S is a semi-open subset of (Y, σ) and O is Ωs-closed subset of (X, τ). Then $f^{-1}(S) \in SO(X)$ and by Proposition 1, $f^{-1}(S) \in SC(X)$. Therefore $sCl(O) \subseteq sCl(f^{-1}(S)) = f^{-1}(S) \subseteq f^{-1}(Int(Cl(S)))$. Hence f is Ωs^*-continuous map.

Theorem 16. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map. If $f(S)$ is a semi-closed subset of (Y, σ), $f(Cl(Int(S))) \subseteq Cl(Int(f(S)))$ for every semi-closed subset S of (X, τ) and $g : (Y, \sigma) \to (Z, \nu)$ is Ωs^*-closed map, then $g \circ f : (X, \tau) \to (Z, \nu)$ is an Ωs^*-closed map.*

Proof. Suppose that S is a semi-closed subset of (X, τ) and O is an Ωs-open subset of (Z, ν) and $g(f(S)) \subseteq O$. Then $g(Cl(Int(f(S)))) \subseteq sInt(O)$. Therefore $g(f(Cl(Int(S)))) \subseteq g(Cl(Int(f(S)))) \subseteq sInt(O)$. Hence $g \circ f$ is Ωs^*-closed map.

Theorem 17. *If $f : (X, \tau) \to (Y, \sigma)$ is an Ωs^*-closed map and $g : (Y, \sigma) \to (Z, \nu)$ is an Ωs-irresolute and pre-semi-open map, then $g \circ f : (X, \tau) \to (Z, \nu)$ is an Ωs^*-closed map.*

Proof. Suppose that S is a semi-closed subset of (X, τ) and O is an Ωs-open subset of (Z, ν) such that $g(f(S)) \subseteq O$. Then $f(S) \subseteq g^{-1}(O)$. By assumption, g is an Ωs-irresolute map; and so $g^{-1}(O)$ is an Ωs-open subset of (Y, σ). Since f is an Ωs^*-closed map, then $f(Cl(Int(S))) \subseteq sInt(g^{-1}(O))$. Hence $g(f(Cl(Int(S)))) \subseteq g(sInt(g^{-1}(O))) = sInt(g(sInt(g^{-1}(O)))) \subseteq sInt(g(g^{-1}(O))) \subseteq sInt(O)$. Therefore, $g \circ f$ is an Ωs^*-closed map.

Theorem 18. *If $f : (X, \tau) \to (Y, \sigma)$ is an Ωs^*-continuous map and $g : (Y, \sigma) \to (Z, \nu)$ is an irresolute map; and $Int(Cl(g^{-1}(S)))) \subseteq g^{-1}(Int(Cl(S)))$ for every semi-open subset S of (Z, ν), then $g \circ f : (X, \tau) \to (Z, \nu)$ is an Ωs^*-continuous map.*

Proof. Let S be a semi-open subset of (Z, ν) and O be an Ωs-closed subset of (X, τ) such that $O \subseteq (g \circ f)^{-1}(S)$. Then $O \subseteq f^{-1}(g^{-1}(S))$ and $g^{-1}(S)$ is a semi-open subset of (Y, σ). Since f is Ωs^*-continuous, then $sCl(O) \subseteq f^{-1}(Int(Cl(g^{-1}(S)))) \subseteq f^{-1}(g^{-1}Int(Cl(S)))) = (g \circ f)^{-1}(Int(Cl(S)))$. Therefore $g \circ f$ is an Ωs^*-continuous map.

The following example shows that the restrictions of Ωs^*-closed and Ωs^*-continuous maps can fail to be Ωs^*-closed and Ωs^*-continuous, respectively.

Example 9. *Let X be an indiscrete space with a nonempty proper subset B. The identity mapping $f : X \to X$ is Ωs^*-closed and hence by Theorem 9 is Ωs^*-continuous.*

First, we prove that $f|_B : B \to X$ is not Ωs^*-closed. Observe that $f(B) = f|_B (B)$ is Ωs-open in X (Proposition 1). Then $f|_B (B) \subseteq f(B)$, where $f(B)$ is Ωs-open *in X and B is a semi-closed in B. But $f\mid_B (Cl_B(Int_B(B))) = f\mid_B (B) = f(B) \nsubseteq sInt(f(B))$ (where $Cl_B(B)$ is the closure of B in B and $Int_B(B)$ is the interior of B in B). Hence $f\mid_B$ is not Ωs^*-closed.*

Second, we prove that $f\mid_B$: $B \to X$ is not Ωs^*-continuous. Since $B \subseteq (f\mid_B)^{-1}(B)$, where B is semi-open in X (Proposition 1) and Ωs-closed in B. But $(f\mid_B)^{-1}(Int(Cl(B))) = f^{-1}(Int(Cl(B))) \cap B \nsupseteq sCl_B(B) = B$ (where $sCl_B(B)$ is the semi-closure of B in B). Hence $f\mid_B$ is not Ωs^*-continuous.

Now, we have the following two theorems

Theorem 19. *If $f : X \to Y$ is an Ωs^*-closed map and B is an open and a semi-closed subset of X, then $f\mid_B$: $B \to Y$ is Ωs^*-closed.*

Proof. Suppose $f\mid_B (S) \subseteq O$, where O is an Ωs-open subset of Y and S is an open and a semi-closed subset of B. Then S is semi-closed in X ([13, Theorem 2.6]) and $f\mid_B (S) = f(S)$. Therefore $f(S) \subseteq O$. Since f is Ωs^*-closed, then $f(Cl(Int(S))) \subseteq sInt(O)$. Now, we prove that $f\mid_B (Cl_B(Int_B(S))) \subseteq f(Cl(Int(S)))$. Since $Cl(E) \cap B = Cl_B(E)$ holds for any set $E \subseteq B$ and $Int(E) \cap B = Int_B(E)$ holds for any $E \subseteq B$ if B is open, then $Cl(Int(S)) \cap B = Cl_B[(Int(S)) \cap B] = Cl_B(Int_B(S))$. Thus, we have $f(Cl(Int(S))) \supseteq f(Cl(Int(S)) \cap B) = f\mid_B (Cl(Int(S)) \cap B) \supseteq f\mid_B (Cl_B(Int_B(S)))$. Therefore, we have that $f\mid_B (Cl_B(Int_B(S))) \subseteq f(Cl(Int(S))) \subseteq sInt(O)$.Hence, $f\mid_B$ is an Ωs^*-closed map.

Theorem 20. *If $f : X \to Y$ is Ωs^*-continuous and B is open and $\Omega-$closed subset of X, then $f\mid_B$: $B \to Y$ is Ωs^*-continuous.*

Proof. Assume $O \subseteq (f\mid_B)^{-1}(S)$, where O is Ωs-closed in B and S is semi-open in Y. Then, we have $O \subseteq f^{-1}(S)$ and O is Ωs-closed relative to X ([16, Theorem 3.4]). Since f is an Ωs^*-continuous map, then $sCl(O) \subseteq f^{-1}(Int(Cl(S)))$. Hence $sCl(O) \cap B \subseteq f^{-1}(Int(Cl(S))) \cap B = (f\mid_B)^{-1}(Int(Cl(S)))$. Since B is open in X, then $sCl(O) \cap B = sCl_B(O)$ [12]. Therefore $sCl_B(O) \subseteq (f\mid_B)^{-1}(Int(Cl(S)))$ and $f\mid_B$: $B \to Y$ is an Ωs^*-continuous map.

6. A characterization of $\Omega - T_{\frac{1}{2}}$ spaces

In the following results, we obtain two properties of $semi-T_{\frac{1}{2}}$ spaces. Furthermore, we offer a characterization of the class of $\Omega - T_{\frac{1}{2}}$ spaces by using the concepts of Ωs^*-closed and Ωs^*-continuous.

Theorem 21. *Let (X, τ) be a topological space.*

(i) For each point $x \in X$, $\{x\}$ is semi-closed or $\Omega s-$open in (X, τ).

(ii) (X, τ) is semi-$T_{\frac{1}{2}}$ if every $\Omega s-$open singleton is semi-open.

Proof. (i) Suppose that a singleton $\{x\}$ is not semi-closed. Then, $X\backslash\{x\}$ is not semi-open; and so the only semi-open set containing $X\backslash\{x\}$ is X . Thus, whenever U is a semi-open set such that $X\backslash\{x\} \subseteq U$, then $U = X$ and $sCl(X\backslash\{x\}) \subseteq X = Int(Cl(U))$ hold; and so $X\backslash\{x\}$ is $\Omega s-$closed. Hence $\{x\}$ is $\Omega s-$open.

(ii)From (i), $\{x\}$ is semi-closed or $\Omega s-$open in (X, τ). By hypothesis $\{x\}$ is semi-closed or semi-open. Then (X, τ) is semi-$T_{\frac{1}{2}}$ [16, Theorem 5.1].

The converse of Theorem 21 (ii) is not true as shown by the following example.

Example 10. *Let $X = \{a, b, c\}$ and $\tau := \{X, \phi, \{a\}\}$ and $F_X := \{X, \phi, \{b, c\}\}$; then it is shown that $SO(X, \tau) := \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $SC(X, \tau) := \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$; and $SGC(X, \tau) := \{X, \phi, \{b\}, \{c\}, \{b, c\}\} = SC(X, \tau)$; $\Omega C(X, \tau) := \{X, \phi, \{b, c\}\}$; $\Omega O(X, \tau) := \{X, \phi, \{a\}\}$; $\Omega_s C(X, \tau) = \Omega_s O(X, \tau) := \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} = P(X)$.*
• One singleton $\{a\}$ is semi-open and two singletons $\{b\}$ and $\{c\}$ are semi-closed. Thus, we conclude that this space (X, τ) is semi-$T_{\frac{1}{2}}$. Indeed, $SGC(X, \tau) = SC(X, \tau)$ holds. Namely, every sg-closed set is semi-closed; and so by definition of semi-$T_{\frac{1}{2}}$ness, this space (X, τ) is semi-$T_{\frac{1}{2}}$. However, there exists a singletons $\{b\}$ such that $\{b\}$ is $\Omega s-$open but $\{b\}$ is not semi-open in (X, τ). Thus, we conclude that the property (=every $\Omega s-$open singleton is semi-open or (open)) is not true for this singleton $\{b\}$ of (X, τ). Therefore, we conclude that the converse of Theorem 21 is not true.

Theorem 22. *If $f : (X, \tau) \to (Y, \sigma)$ is an Ωs^*-continuous map for any space (Y, σ), then the space (X, τ) is an $\Omega - T_{\frac{1}{2}}$ space.*

Proof. Let O be an $\Omega s-$closed subset of X and Y be the set X with the topology $\sigma = \{Y, O, Y\backslash O, \phi\}$. Let $f : X \to Y$ be the identity map. By assumption, f is an Ωs^*-continuous map. Since O is $\Omega s-$closed in X , open and closed in Y, and $O \subseteq f^{-1}(O)$, then $sCl(O) \subseteq f^{-1}(Int(Cl(O)) = f^{-1}(O) = O$. Hence, O is semi-closed in X. Therefore, the space Y is $\Omega - T_{\frac{1}{2}}$.

Theorem 23. *If $f : (X, \tau) \to (Y, \sigma)$ is an Ωs^*-closed map for any space (X, τ), then the space (Y, σ) is an $\Omega - T_{\frac{1}{2}}$ space.*

Proof. Let O be an $\Omega s-$open subset of Y and X be the set Y with the topology $\tau = \{X, O, X\backslash O, \phi\}$. Let $f : X \to Y$ be the identity map. By assumption, f is Ωs^*-closed. Since O is $\Omega s-$open in Y, open and closed in X, and $f(O) \subseteq O$, it follows that $O = f(O) = f(Cl(Int(O))) \subseteq sInt(O)$. Hence, O is semi-open in Y. Therefore, the space Y is $\Omega - T_{\frac{1}{2}}$.

The converse of both Theorem 22 and 23 is not true as shown by the following example

Example 11. *Let* $X = \{a, b, c\}$ *and* $\tau := \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ *and* $F_X := \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}$; *then it is shown that* $SO(X, \tau) = SC(X, \tau) = P(X)$; $\Omega_s C(X, \tau) = \tau$; *and* $\Omega_s O(X, \tau) = F_X$. *The space* (X, τ) *is* $\Omega - T_{\frac{1}{2}}$. *Now, define the map* $f : (X, \tau) \to (X, \tau)$ *to be:* $f(a) = a$, $f(b) = c$ *and* $f(c) = b$. f *is not* Ωs^**-continuous. Indeed, we have* $\{b\} \subseteq f^{-1}(\{c\})$, *where* $\{b\} \in \Omega_s C(X, \tau)$ *and* $\{c\} \in SO(X, \tau)$, *but* $sCl(\{b\}) = \{b\} \nsubseteq f^{-1}(Int(Cl(\{c\}))) = \phi$. *We conclude that* f *is not an* Ωs^**-continuous map and the converse of Theorem 22 is not true.*

Furthermore, f *is not* Ωs^**- closed. Indeed, we have* $f(\{b\}) \subseteq \{c\}$,*where* $\{b\} \in SC(X, \tau)$ *and* $\{c\} \in \Omega_s O(X, \tau)$, *but* $f(Cl(Int(\{b\}))) = \{b, c\} \nsubseteq \{c\} = sInt(\{c\})$. *We conclude that* f *is not an* Ωs^**-closed map and the converse of Theorem 23 is not true.*

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