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THE INFLUENCE OF PARTIALLY S-EMBEDDED SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

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Abstract. Let G be a finite group and H a subgroup of G, then H is said to be s-permutable (respectively, s-semipermutable) in G if $HP = PH$ hold for every Sylow subgroup P (respectively, with $(|P|, |H|) = 1$) of G. Let $H_{\bar{s}G}$ be the subgroup of H generated by all those subgroups which are s-semipermutable in G, then we say that H is partially S-embedded in G if G has a normal subgroup T such that HT is s-permutable in G and $T \cap H \leq H_{\bar{s}G}$. In this paper, some new criteria about the p-nilpotency and supersolvability of a finite group G are obtained. A series of known results in the literature are unified and generalized.

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1. Introduction

In this paper, all groups considered are finite and G stands for a finite group. Let $\mathcal{F}$ be a formation, $\mathcal{U}$ and $\mathcal{N}_p$ denote the class of all supersolvable groups and p-nilpotent groups, respectively. $G^{\mathcal{F}}$ stands for the $\mathcal{F}$-residual of G, that is, the intersection of all normal subgroups N_i of G such that $G/N_i \in \mathcal{F}$.

The relations between the generalized normal subgroups and the structure of a group is always a question of particular interest. Following Kegel [12], a subgroup H is said to be s-permutable (or s-quasinormal [4]) in G, if $HP = PH$ for every Sylow subgroup P of G. On the other hand, Wang in [20] introduced the concept of c-normal subgroup from the idea of the supplement subgroup: a subgroup H is said to be c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. These two kind of subgroups have been investigated extensively by many scholars. Recently, Guo et al [8] integrated these two concepts and introduced that: a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. As another generation of the s-permutable subgroup, Chen in [3] introduced that: a subgroup H of a group G is said to be s-semipermutable (or s-seminormal) in G if $PH = HP$ holds for every Sylow subgroup P of G with $(|P|, |H|) = 1$. By assuming that some subgroups of G satisfy the S-embedded property or s-semipermutablity, many interesting results have been derived (see [8], [9], [24], [25] etc.). Motivated by the above research, we now introduce the following new concept, which can cover the s-permutable, s-semipermutable and S-embedded subgroups properly.

Definition 1. *A subgroup H of G is said to be partially S-embedded in G, if G has a normal subgroup T such that HT is s-permutable in G and $H \cap T \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ is generated by all those subgroups of H which are s-semipermutable in G.*

It is easy to see that $H_{\bar{s}G}$ is an s-semipermutable subgroup of G. Besides that, from our Definition 1, we know every S-embedded subgroup and s-semipermutable subgroup of G is partially S-embedded in G. In general, a partially S-embedded subgroup of G need not to be S-embedded or s-semipermutable in G. For instance:

Example 1. *Let $G = S_5$ be the symmetric group of degree 5. Since $H = S_4$ permutes with every Sylow 5-subgroup of G, H is s-semipermutable and thus partially S-embedded in G. Since H and $H \cap A_5 = A_4$ are not subnormal in G, they are not s-permutable in G. Hence from the fact that the only nontrivial normal subgroups of G are A_5 and G itself, we know $H = S_4$ is not S-embedded in G.*

Example 2. *Let $G = S_5$, $K = \langle(12)\rangle$ and $T = A_5$. Since $T \trianglelefteq G$, $KT = G$ and $K \cap T = 1 \leq K_{\bar{s}G}$, K is partially S-embedded in G. But the fact $K\langle(12345)\rangle \neq \langle(12345)\rangle K$ implies that K is not s-semipermutable in G.*

In this paper, some results about the influence of partially S-embedded subgroups on the structure of a finite group are given, a series of known results are generalized.

2. Preliminaries

Lemma 1. *([12]) Suppose that H is an s-permutable subgroup of G and $N \trianglelefteq G$.*

(1) If $K \leq G$, then $H \cap K$ is s-permutable in K.

(2) HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.

(3) H is subnormal in G.

(4) If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.

Lemma 2. *([25]) Let G be a group and $H \leq K \leq G$.*

(1) If H is s-semipermutable in G, then H is s-semipermutable in K.

(2) Suppose that N is normal in G, and H is a p-group. If H is s-semipermutable in G, then HN/N is s-semipermutable in G/N.

(3) If H is an s-semipermutable and K a quasinormal subgroup of G, then $H \cap K$ is s-semipermutable in G.

Now, we prove that:

Lemma 3. *Suppose that H is a partially S-embedded subgroup of G.*

(1) If $H \leq K \leq G$, then H is partially S-embedded in K.

(2) Let H be a p-group and $N \trianglelefteq G$. If $N \leq H$ or $(p, |N|) = 1$, then HN/N is partially S-embedded in G/N.

Proof. Suppose that $T \trianglelefteq G$, HT is s-permutable in G and $H \cap T \leq H_{\overline{s}G}$.

(1) Clearly, $K \cap T$ is a normal subgroup of K. By Lemmas 1 and 2, we know that $H(K \cap T) = K \cap HT$ is s-permutable in K and $H \cap (K \cap T) = H \cap T \leq H_{\overline{s}G} \leq H_{\overline{s}K}$. Hence, H is partially S-embedded in K.

(2) It is easy to see that $TN/N \trianglelefteq G/N$ and $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. If $N \leq H$, then $H/N \cap TN/N = (H \cap T)N/N \leq H_{\overline{s}G}N/N$. If N is a p'-group, then

$$|H \cap TN| = \frac{|H| \cdot |TN|_p}{|HTN|_p} = \frac{|H| \cdot |T|_p}{|HT|_p} = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, we also conclude that $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N \leq H_{\overline{s}G}N/N$. By Lemma 2, we know that $H_{\overline{s}G}N/N$ is s-semipermutable in G/N. Hence, HN/N is partially S-embedded in G/N in any case.

Lemma 4. *([25, Lemma 3]) Let H be a subnormal p-subgroup of G. If H is s-semipermutable in G, then H is s-permutable in G.*

The following result is well known

Lemma 5. *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$. If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*

Lemma 6. *([5, A, Lemma 1.2]) Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(a) $U \cap VW = (U \cap V)(U \cap W)$;

(b) $UV \cap UW = U(V \cap W)$.

3. Main results

Theorem 7. *Let P be a Sylow p-subgroup of a group G, where $p \in \pi(G)$ and $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every maximal subgroup of P is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) P is not cyclic and G is not a non-abelian simple group.

By Lemma 5, we may assume that P is not cyclic. Let P_1 be a maximal subgroup of P, by hypothesis we know P_1 is partially S-embedded in G. Then there exists a normal subgroup K_1 of G such that P_1K_1 is an s-permutable subgroup of G and $P_1 \cap K_1 \leq (P_1)_{\bar{s}G}$. If G is a non-abelian simple group, then $K_1 = 1$ or G. First assume that $K_1 = 1$, in this case, $P_1 = P_1K_1$ is s-permutable in G. Hence P_1 is a proper subnormal subgroup of G, which is a contradiction. Thus $K_1 = G$ and therefore $P_1 = P_1 \cap K_1 = (P_1)_{\bar{s}G}$ is s-semipermutable in G. The above statements hold for every maximal subgroup of P. In other words, all maximal subgroups of P are s-semipermutable in G.

Let H be any nontrivial subgroup of P, we consider $N_G(H)$. Suppose that $S_1 \in Syl_p(N_G(H))$ and $Q_1 \in Syl_q(N_G(H))$ for any prime $q \neq p$. Let Q be a Sylow q-subgroup of G containing Q_1, then every maximal subgroup of P is permutable with Q. Since P is not cyclic, $P = P_1P_2$ for some maximal subgroups P_1 and P_2 of P. Thus $PQ = P_1P_2Q = QP_1P_2 = QP$ is a proper Hall subgroup of G, as PQ is solvable. It is easy to see that PQ satisfies the hypothesis of the theorem. Then the minimal choice of G implies that PQ is p-nilpotent. Hence $Q \trianglelefteq PQ$ and $Q_1 = Q \cap N_{PQ}(H) \trianglelefteq N_{PQ}(H)$. We conclude that $HQ_1 = H \times Q_1$ for any Sylow q-subgroup Q_1 of $N_G(H)$ with $q \neq p$. Hence $N_G(H)$ is p-nilpotent. From the Frobenius Theorem [10, IV, Theorem 5.8], we know G is p-nilpotent. This contradiction implies that G is not a non-abelian simple group.

(2) G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G and M/N a maximal subgroup of PN/N. It is easy to see that $M = P_1N$ for some maximal subgroup P_1 of P and $P \cap N = P_1 \cap N$ is a Sylow p-subgroup of N. Since P_1 is partially S-embedded in G, there exists a normal subgroup K of G such that P_1K is s-permutable in G and $P_1 \cap K \leq (P_1)_{\bar{s}G}$. Clearly, KN/N is a normal subgroup of G/N and $P_1N/N \cdot KN/N = P_1KN/N$ is s-permutable in G/N. Moreover, since $P_1 \cap N$ is a Sylow p-subgroup of N, $|(P_1 \cap N)(K \cap N)|_p = |P_1 \cap N| = |N|_p = |N \cap P_1K|_p$ and

$$|P_1K \cap N|_{p'} = \frac{|P_1K|_{p'} \cdot |N|_{p'}}{|P_1KN|_{p'}} = \frac{|K|_{p'} \cdot |N|_{p'}}{|KN|_{p'}} = |K \cap N|_{p'} = |(P_1 \cap N)(K \cap N)|_{p'}.$$

This implies that $(P_1 \cap N)(K \cap N) = P_1K \cap N$. Thus by Lemma 6, we have $P_1N \cap KN = (P_1 \cap K)N$. Then it follows from Lemma 2 that $P_1N/N \cap KN/N = (P_1 \cap K)N/N \leq (P_1)_{\overline{s}G}N/N \leq (P_1N/N)_{\overline{s}(G/N)}$, and so M/N is partially S-embedded in G/N. Therefore, G/N satisfies the hypothesis and so it is p-nilpotent by the minimal choice of G. Since the class of all p-nilpotent groups formed a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(3) $O_{p'}(G) = O_p(G) = 1$ and N is not p-nilpotent.

If $O_{p'}(G) \neq 1$, then by (2) we know $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent. Hence G is p-nilpotent, a contradiction. If $O_p(G) \neq 1$, then $N \leq O_p(G)$ is an elementary abelian p-group. Since $\Phi(G) = 1$, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. From the unique minimal normality of N, we can easily deduce that $N = O_p(G)$. Since $P = N(P \cap M)$ and $N \cap M = 1$, $P \cap M$ is a Sylow p-subgroup of M and there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$ and $P = NP_1$. Since P_1 is partially S-embedded in G, there exists some normal subgroup T of G such that P_1T is s-permutable in G and $P_1 \cap T \leq (P_1)_{\overline{s}G}$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. It follows from Lemma 1(3) that $P_1 \leq O_p(G) = N$ and so $P = P_1N = N$ is a minimal normal subgroup of G. Since $N_G(P_1) \geq O^p(G)$ by Lemma 1(4) and $P_1 \trianglelefteq P$, P_1 is a proper normal subgroup of G contained in $P = O_p(G)$, a contradiction. Thus, $T \neq 1$ and so $N \leq T$. In this case, $P_1 \cap T = (P_1)_{\overline{s}G} \cap T$ is s-semipermutable in G. Therefore, for any Sylow q-subgroup Q of G with $q \neq p$, we have

$$N \cap P_1 = N \cap P_1 \cap T = N \cap (P_1 \cap T)Q \trianglelefteq (P_1 \cap T)Q.$$

Hence $Q \leq N_G(N \cap P_1)$ and then $O^p(G) \leq N_G(N \cap P_1)$. Since $N \cap P_1 \trianglelefteq P$, it is normal in G. Thus $N \cap P_1 = 1$ and $|N| = p$. Let C/N be the normal p-complement of G/N, then N is a cyclic Sylow p-subgroup of C. By Lemma 5, C is p-nilpotent and the normal p-complement of C is also the normal p-complement of G, a contradiction.

If N is p-nilpotent, then $N_{p'}$ *char* $N \trianglelefteq G$, so $N_{p'} \leq O_{p'}(G) = 1$. Thus N is a p-group and so $N \leq O_p(G) = 1$, a contradiction too.

(4) $G = PN$.

By Lemma 3, we know PN satisfies the hypothesis of the theorem. Therefore, PN is p-nilpotent if $PN < G$. It follows that N is p-nilpotent, which contradicts with (3). Hence, we have $G = PN$ and $N = O^p(G)$.

(5) The final contradiction.

Since N is non-solvable, $N = S_1 \times S_2 \times \cdots \times S_k$ is a direct product of some isomorphic non-abelian simple groups S_i. By (1) and (4), we know $N < G$ and $P \cap N < P$. Thus there exists some maximal subgroup P_1 of P such that $S_p = P \cap S_1 \leq P_1$, where S_p is a Sylow p-subgroup of S_1. By hypothesis, there exists a normal subgroup T of G such that P_1T is s-permutable in G and $P_1 \cap T \leq (P_1)_{\overline{s}G}$.

If $T = 1$, then P_1 is s-permutable in G and so $O_p(G) \neq 1$, this contradicts with (3). Thus $T \neq 1$ and the uniqueness of N implies that $N \leq T$. If $P_1 \cap T = 1$, then $|T|_p \leq p$. Hence by Lemma 5, we know T is p-nilpotent and so N is p-nilpotent. This contradiction shows that $P_1 \cap T \neq 1$ and $P_1 \cap T = (P_1)_{\overline{s}G} \cap T$ is s-semipermutable in G. Then for any prime divisor q of $|G|$ different from p and any Sylow q-subgroup Q of G, $(P_1 \cap T)Q = Q(P_1 \cap T)$ is a subgroup of G. Since

$$|Q \cap P_1T| = \frac{|Q| \cdot |P_1T|_q}{|QP_1T|_q} = \frac{|Q| \cdot |T|_q}{|QT|_q} = |Q \cap T| = |(Q \cap P_1)(Q \cap T)|$$

and $(Q \cap P_1)(Q \cap T) \subseteq Q \cap P_1T$, $Q \cap P_1T = (Q \cap P_1)(Q \cap T)$. By Lemma 6, we have $QP_1 \cap QT = Q(P_1 \cap T)$. Therefore, $N \cap P_1Q = N \cap (P_1Q \cap TQ) = N \cap (P_1 \cap T)Q$. This implies that $S_1 \cap (P_1 \cap T) = S_1 \cap P_1 = S_p$ is a Sylow p-subgroup and $S_1 \cap Q$ is a Sylow q-subgroup of S_1. Thus for any prime $q \neq p$, $S_1 \cap (P_1 \cap T)Q$ is a Hall $\{p, q\}$-subgroup of S_1. Since N is non-abelian and $(|N|, p-1) = 1$, $p = 2$. Then for any prime divisor $q \neq 2$ of $|S_1|$, the non-abelian simple group S_1 has a Hall $\{2, q\}$-subgroup, which contradicts with [14, Lemma 2.6]. This contradiction completes the proof of the theorem.

If we replace the condition that "$(|G|, p-1) = 1$" with "$N_G(P)$ is p-nilpotent" in Theorem 7, we can also get the following similar result:

Theorem 8. *Let p be a prime divisor and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is partially S-embedded in G, then G is p-nilpotent.*

Proof. If $p = min\pi(G)$, then by Theorem 7 we know that G is p-nilpotent. Hence we only need to consider the case that $p \neq min\pi(G)$ (and so p is an odd prime). Assume that the result is false and let G be a counterexample of minimal order. Then we have:

(1) Every proper subgroup of G containing P is p-nilpotent.

Let M be a proper subgroup of G containing P. Since $N_M(P) \leq N_G(P)$ is p-nilpotent, by Lemma 3 we know M satisfies the hypothesis of the theorem. Thus, the minimal choice of G implies that M is p-nilpotent.

(2) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$, then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Let $T/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$, then $T = P_1O_{p'}(G)$ holds for some maximal subgroup P_1 of P. By Lemma 3, we know $P_1O_{p'}(G)/O_{p'}(G)$ is partially S-embedded in $G/O_{p'}(G)$. This shows that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Then $G/O_{p'}(G)$ is p-nilpotent by induction, which implies that G is also p-nilpotent, a contradiction. This contradiction shows that $O_{p'}(G) = 1$.

(3) $G = PQ$ is solvable and $1 < O_p(G) < P$, where Q is a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson's theorem [17, Theorem 10.4.1], there exists a nontrivial characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose H satisfying that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P containing H. Obviously, $N_G(P) \leq N_G(H)$. Then by (1), $N_G(H) = G$. Therefore, we have $H \leq O_p(G) < K$. Now by the Thompson's theorem again, we see that $G/O_p(G)$ is p-nilpotent, and so G is p-solvable. By [6, VI, Theorem 3.5], there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G, where q is a prime divisor of $|G|$ which is different from p. If $PQ < G$, then PQ is p-nilpotent by (1). This implies that $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $G = PQ$ and (3) holds.

(4) G has a unique minimal normal subgroup N such that $G = [N]M$, where M is a maximal subgroup of G and $N = O_p(G) = F(G)$.

Let N be a minimal normal subgroup of G. Then by (2) and (3), N is an elementary abelian p-group and $N \leq O_p(G)$. It is easy to see that G/N satisfies the hypothesis of the theorem. Then the minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups formed a saturated formation, N is the unique minimal normal subgroup of G and $N \not\leq \Phi(G)$. Thus, there exists a maximal subgroup M of G such that $G = MN$. Since $O_p(G) \leq F(G) \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$, we can deduce that $N = O_p(G) = F(G)$.

(5) N is a cyclic group of order p.

Let M_p be a Sylow p-subgroup of M, then $P = NM_p$ and $N \cap M_p = 1$. Let P_1 be a maximal subgroup of P containing M_p. If $P_1 = 1$, then $|N| = |P| = p$. Now suppose that $P_1 \neq 1$. By hypothesis, there exists some normal subgroup K of G such that P_1K is s-permutable in G and $P_1 \cap K \leq (P_1)_{\bar{s}G}$. If $K = 1$, then $P_1 = P_1K$ is s-permutable in G which implies that $P_1 \leq O_p(G) = N$. Therefore, we have $P = NP_1 = N$, which is contradict with (3). Thus, $K \neq 1$ and then $N \leq K$. In this case, $P_1 \cap K = (P_1)_{\bar{s}G} \cap K$ is s-semipermutable in G and

$$N \cap P_1 = N \cap P_1 \cap K = N \cap (P_1 \cap K)Q \trianglelefteq (P_1 \cap K)Q.$$

Hence, we conclude that $Q \leq N_G(N \cap P_1)$. Since $P_1 \cap N \trianglelefteq P$, it is normal in G. Thus, the minimal normality of N implies that $P_1 \cap N = 1$ and so $|N| = p$.

(6) The final contradiction.

By (4) and (5), we know $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic with some subgroup of $Aut(P)$, which is a cyclic group of order $p - 1$. Hence M and in particularly, Q is a cyclic group. It follows form [17, Theorem 10.1.9] that G is q-nilpotent, in other words, $P \trianglelefteq G$. Then by hypothesis, $N_G(P) = G$ is p-nilpotent. This contradiction completes the proof of the theorem.

Next, by using the partially S-embedded properties of some subgroups, we give out some new criteria for the supersolvability of a group G.

Theorem 9. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if there exists a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of E is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Suppose that the result is false and let G be a counterexample with $|G||E|$ minimal. Then we have:

(1) E is solvable and $Q \trianglelefteq G$, where $q = max\pi(E)$ and $Q \in Syl_q(E)$.

Let $p = min\pi(E)$ and P a Sylow p-subgroup of E. If P is cyclic, then E is p-nilpotent by Lemma 5. Now suppose that P is not cyclic and P_1 is a maximal subgroup of P. Then by hypothesis, P_1 is partially S-embedded in G. Thus it is partially S-embedded in E by Lemma 3. From Theorem 7, we know E is p-nilpotent. Let K be the normal p-complement of E. By hypothesis and Lemma 3, we can deduce that every maximal subgroup of any non-cyclic Sylow subgroup of K is partially S-embedded in K. Thus, we can conclude that E is a Sylow tower group of supersolvable type and so it is solvable. Let q be the largest prime divisor and Q a Sylow q-subgroup of E. Since Q $char$ $E \trianglelefteq G$, Q is normal in G.

(2) There is a unique minimal normal subgroup N of G contained in E, $G/N \in \mathcal{F}$ and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G contained in E. Since E is solvable, N is an elementary abelian p-group, where p is a prime. Obviously, $(G/N)/(E/N) \cong G/E \in \mathcal{F}$. Let T/N be a noncyclic Sylow r-subgroup of E/N and T_1/N a maximal subgroup of T/N, where r is a prime divisor of $|E/N|$. If $r = p$, then T is a noncyclic Sylow p-subgroup of E and T_1 is a maximal subgroup of T containing N. By hypothesis, T_1 is partially S-embedded in G. So T_1/N is partially S-embedded in G/N by Lemma 3. Now suppose that $r \neq p$. In this case there exists a Sylow r-subgroup R of E such that $T = RN$. Let $R_1 = R \cap T_1$, then R_1 is a maximal subgroup of R and $T_1 = R_1N$. Therefore, R_1 is partially S-embedded in G and so T_1/N is partially S-embedded in G/N. This shows that $(G/N, E/N)$ satisfies the hypothesis of the theorem. Then the minimal choice of G implies that $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in E and $N \nleq \Phi(G)$. Therefore, $\Phi(G) = 1$.

(3) $N = Q = F(E)$ is not a cyclic group, $G = [N]M$ hold for some maximal subgroup M of G.

Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = [N]M$. Since $C = C_E(N) = C_G(N) \cap E \trianglelefteq G$, $(C \cap M)^G = (C \cap M)^{NM} = (C \cap M)^M = C \cap M$, i.e., $C \cap M$ is a normal subgroup of G. It follows that $C \cap M = 1$ and $C = N$. Since

$N \leq O_q(E) \leq F(E) \leq F(G) \leq C_G(N)$, $N = F(E) = Q$. In view of (2), $G/N \in \mathcal{F}$. By [18, Lemma 2.16], we may assume that N is not cyclic.

(4) The final contradiction.

Let M_q be a Sylow q-subgroup of M and $G_q = NM_q$. Since $G = [N]M$ and N is not cyclic, G_q is a noncyclic Sylow q-subgroup of G. Let Q_1 be a maximal subgroup of G_q containing M_q and $N_1 = N \cap Q_1$, then $N_1 \trianglelefteq G_q$. Since $|N : N_1| = |N : N \cap Q_1| = |NQ_1 : Q_1| = |G_q : Q_1| = q$, N_1 is a maximal subgroup of N. By hypothesis, there exists a normal subgroup K of G such that N_1K is s-permutable in G and $N_1 \cap K \leq (N_1)_{\bar{s}G}$. In view of (2), we see that $N \cap K = 1$ or $N \leq K$. If $N \cap K = 1$, then $N_1 = N_1(N \cap K) = N \cap N_1K$ is s-permutable in G by Lemma 1(2). If $N \leq K$, then $N_1 = N_1 \cap K = (N_1)_{\bar{s}G}$ is s-semipermutable in G. By Lemma 4, we also have that N_1 is s-permutable in G. Consequently, by Lemma 1(4), $N_G(N_1) \geq O^q(G)$. On the other hand, $N_1 = N \cap Q_1 \trianglelefteq G_q$. This implies that $N_1 \trianglelefteq G$. Thus $N_1 = 1$ and $|N| = q$, which contradicts with (3). This contradiction completes the proof of the theorem.

From our Theorem 9, when $\mathcal{F} = \mathcal{U}$ we have:

Corollary 10. *A group G is supersolvable if and only if there is a normal subgroup E such that G/E is supersolvable, and every maximal subgroup of any noncyclic Sylow subgroup of E is partially S-embedded in G.*

We use $F^*(G)$ to denote the generalized Fitting subgroup of G, i.e., $F^*(G) = F(G)E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the layer of G.

Theorem 11. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$, and every maximal subgroup of any non-cyclic Sylow subgroup of $F^*(E)$ is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. Assume that the result is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$ and $F^* = F^*(E)$. We use p to denote the minimal prime divisor of $|F^*(E)|$ and let P be a Sylow p-subgroup of $F^*(E)$.

If P is cyclic, then by [10, IV, Theorem 2.8], we know that $F^*(E)$ is p-nilpotent. Now we assume that P is not cyclic, by hypothesis and Lemma 3, we have every maximal subgroup of P is partially S-embedded in $F^*(E)$. By Corollary 10, we can also deduce that $F^*(E)$ is p-nilpotent. Therefore, we know that $F^* = F$ is solvable. If $F = E$, then $G \in \mathcal{F}$ by Theorem 9, which contradict with the choice of G. Hence we may assume that $F^* = F \neq E$. Now by [11, X, Theorem 13.11], we have $C_E(F) = C_E(F^*) \leq F$. Since $F^* = F$ is a solvable normal subgroup of G, by hypothesis and Lemma 4 we can easily deduce that every maximal subgroup of any non-cyclic Sylow subgroup of F^* is S-embedded in G. Now, from [8, Theorem D], we can conclude that $G \in \mathcal{F}$, as required.

From the partially S-embedded properties of some subgroups, we can also characterize the nilpotency of a finite group G:

Theorem 12. *A group G is nilpotent if and only if for every prime $p \in \pi(G)$ and every Sylow p-subgroup P of G, $N_G(P)/C_G(P)$ is a p-group and every maximal subgroup of P is partially S-embedded in G.*

Proof. The necessity is obvious, we need to prove only the sufficiency. By Corollary 10, we know G is supersolvable. Let q be the largest prime divisor and Q a Sylow q-subgroup of G, then clearly we have $Q \trianglelefteq G$.

Let N be a minimal normal subgroup of G contained in Q and $\overline{P}$ a Sylow p-subgroup of $\overline{G} = G/N$, then there exists a Sylow p-subgroup P of G such that $\overline{P} = PN/N$. Obviously, $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$ and $C_{\overline{G}}(\overline{P}) \geq C_G(P)N/N$. Hence $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is a p-group. Let R_1/N be a maximal subgroup of PN/N. If $p = q$, then $N \leq P$ and R_1 is a maximal subgroup of P. By hypothesis, R_1 is partially S-embedded in G, so R_1/N is partially S-embedded in G/N. If $p \neq q$, then $R_1 = R_1 \cap PN = (R_1 \cap P)N$ and $R_1 \cap P$ is a maximal subgroup of P. By hypothesis, $R_1 \cap P$ is partially S-embedded in G, consequently $R_1/N = (R_1 \cap P)N/N$ is partially S-embedded in G/N by Lemma 3. This shows that G/N satisfies the hypothesis of the theorem. Thus G/N is nilpotent by induction. Since the class of all nilpotent groups formed a saturated formation, N is a unique minimal normal subgroup of G contained in Q and $\Phi(G) = 1$. Hence there exists a maximal subgroup M such that $G = NM$. Since G is solvable, N is an elementary abelian group and so $N \cap M = 1$. Then we have $Q = Q \cap NM = N(Q \cap M)$ and $Q \cap M \leq Q \leq F(G) \leq C_G(N)$. Thus $(Q \cap M)^G = (Q \cap M)^{MN} = Q \cap M$, i.e., $Q \cap M \trianglelefteq G$. Therefore, we conclude that $Q \cap M = 1$, $N = Q$ and $Q \leq C_G(Q)$. The condition $N_G(Q)/C_G(Q)$ is a q-group implies that $N_G(Q) = C_G(Q) = G$. Consequently, $Q \leq Z(G)$. Since G/Q is nilpotent, G is nilpotent as well, as required.

4. Some applications

Our Theorems 7, 9 and 11 generalized main results of a large number of papers. For example, since all s-permutable (or π-quasinormal) subgroups and c-normal subgroups of G are partially S-embedded in G, by Theorems 9 and 11 we have

Corollary 13. *([19]) Let G be a finite group with the property that maximal subgroups of Sylow subgroups are π-quasinormal in G for $\pi = \pi(G)$. Then G is supersolvable.*

Corollary 14. *([2]) If G/H is supersolvable and all maximal subgroups of any Sylow subgroup of H are π-quasinormal in G, then G is supersolvable.*

Corollary 15. *([1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of H are π-permutable in G, then $G \in \mathcal{F}$.*

Corollary 16. *([16]) Assume that G is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is π-quasinormal in G. Then G is supersolvable.*

Corollary 17. *([2]) Let G be a solvable group. If G/H is supersolvable and all maximal subgroups of any SyIow subgroup of $F(H)$ are π-quasinormal in G, then G is supersolvable.*

Corollary 18. *([15]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$, and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are π-quasinormal in G, then $G \in \mathcal{F}$.*

Corollary 19. *([20]) Let G be a finite group. Suppose P_1 is c-normal in G for every Sylow subgroup P of G and every maximal subgroup P_1 of P. Then G is supersolvable.*

Corollary 20. *([13]) Let G be a solvable group. If H is a normal subgroup of G such that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ are c-ncrmal in G, then G is supersolvable.*

Corollary 21. *([21]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(E)$ are c-normal in G, then $G \in \mathcal{F}$.*

Corollary 22. *([22]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(E)$ are c-normal in G, then $G \in \mathcal{F}$.*

Following [7], a subgroup H is said to be nearly s-normal in G, if there exists a normal subgroup N of G such that $HN \trianglelefteq G$ and $H \cap N \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. From the definition we know a nearly s-normal subgroup of G is S-embedded in G, then it is partially S-embedded in G and we have

Corollary 23. *([7]) A group G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and every maximal subgroup of every noncyclic Sylow subgroup of H is nearly s-normal in G.*

Corollary 24. *([9]) A group G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and all maximal subgroups of every noncyclic Sylow subgroup of H are S-embedded in G.*

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References

[1] M. Asaad, *On maximal subgroups of Sylow subgroups of finite groups*, Comm. Algebra 26, (1998), 3647-3652.

[2] M. Asaad, M. Ramadan and A. Shaalan, *Influence of π-quasinormality on maximal subgroups of Sylow subgroups of Fitting subgroup of a finite group*, Arch. Math.(Basel) 56, (1991), 521-527.

[3] Z. M. Chen, *On a theorem of Srinivasan*, J. of Southwest Normal Univ.(Nat Sci) 12, (1987), 1-4.

[4] W. E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z. 82, (1963), 125-132.

[5] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, 1992.

[6] D. Gorenstein, *Finite Groups*, Chelsea, New York, 1968.

[7] W. B. Guo, Y. Wang and L. Shi, *Nearly s-normal subgroups of finite group*, J. Alg. Disc. Struc. 6(2), (2008), 95-106.

[8] W. B. Guo, K. P. Shum and A. N. Skiba, *On solubility and supersolubility of some classes of finite groups*, Sci. China (Ser. A) 52(2), (2009), 272-286.

[9] W. B. Guo, Y. Lu and W. J. Niu, *S-embedded subgroups of finite groups*, Algebra Log. 49(4), (2010), 293-304.

[10] B. Huppert, *Endliche Gruppen Vol. I*, Springer, Berlin, 1967.

[11] B. Huppert and N. Blackburn, *Finite Groups III*, Springer, Berlin, 1982.

[12] O. H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math. Z. 78, (1962), 205-221.

[13] D. Y. Li and X. Y. Guo, *The influence of c-normality of subgroups on the structure of finite groups II*, Comm. Algebra 26, (1998), 1913-1922.

[14] Y. M. Li and X. H. Li, *Z-permutable subgroups and p-nilpotency of finite groups*, J. Pure Appl. Algebra 202, (2005), 72-81.

[15] Y. M. Li, Y. M. Wang and H. Q. Wei, *The influence of π-quasinormality of some subgroups of a finite group*, Arch. Math. 81, (2003), 245-252.

[16] M. Ramadan, *Influence of normality on maximal subgroups of Sylow subgroups of a finite group*, Acta Math. Hung. 59, (1992), 107-110.

[17] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, New York, 1993.

[18] A. N. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra 315, (2007), 192-209.

[19] S. Srinivasan, *Two sufficient conditions for supersolvability of finite groups*, Israel J. Math. 35, (1980), 210-214.

[20] Y. M. Wang, *C-normality of groups and its properties*, J. Algebra 180, (1996), 954-965.

[21] H. Q. Wei, *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups*, Comm. Algebra 29, (2001), 2193-2200.

[22] H. Q. Wei, Y. M. Wang and Y. M. Li, *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups II*, Comm. Algebra 31, (2003), 4807-4816.

[23] H. Q. Wei and Y. M. Wang, *The c-supplemented property of finite groups*, P. Edinburgh Math. Soc. 50, (2007), 493-508.

[24] Q. H. Zhang, *On s-smipermutability and abnormality in finite groups*, CommAlgebra 27, (1999), 4515-4524.

[25] Q. H. Zhang and L. F. Wang, *The influence of s-semipermutable properties of subgroups on the structure of finite groups*, Acta Math. Sin. 48, (2005), 81-88.

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