Acta Universitatis Apulensis No. 38/2014
ISSN: 1582-5329 pp. 109-118

ON A SUBCLASS OF MEROMORPHIC FUNCTION WITH FIXED SECOND COEFFICIENT

T. Rosy and S. Sunil Varma

Abstract. In this paper we introduce a new subclass of meromorphic function with fixed second coefficient defined by Fox-Wright's generalized hypergeometric function. We obtain coefficient estimates, extreme points, growth and distortion theorems, radii of meromorphically starlikeness and convexity for this new subclass.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: meromorphic functions, Hadamard product, fixed second coefficient, coefficient inequalities, radii of meromorphically starlikeness and convexity.

1. Introduction

We denote by Σ the class of functions of the form

$$f(z) = \frac{1}{z} + \Sigma_{n=1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the punctured unit disk

$$\Delta^* := \{z \in \mathbb{C} /\ 0 < |z| < 1\}.$$

Let Σ_P denote the class of functions of the form (1) with $a_n \geq 0$ i.e.

$$f(z) = \frac{1}{z} + \Sigma_{n=1}^{\infty} a_n z^n,\ a_n \geq 0. \tag{2}$$

A function $f \in \Sigma$ is said to be meromorphically starlike of order α if

$$-Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$

and meromorphically convex of order α if

$$-Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

We denote the class of meromorphically starlike functions and the class of meromorphically convex functions by $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$ respectively. Various subclasses of Σ have been defined and studied by various authors (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17]).

The Hadamard product between $f \in \Sigma$ given by (1.1) and $g(z) = \frac{1}{z} + \Sigma_{n=1}^{\infty} b_n z^n \in \Sigma$ is defined as

$$(f * g)(z) = \frac{1}{z} + \Sigma_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

For positive real parameters $\alpha_1, A_1, \dots, \alpha_l, A_l, \beta_1, B_1, \dots, \beta_m, B_m$ $(l, m \in \mathbb{N} = \{1, 2, 3, \dots\})$ such that $1 + \Sigma_{k=1}^{m} B_k - \Sigma_{k=1}^{l} A_k \geq 0, z \in \{z \in \mathbb{C}/0 < |z| < 1\}$ the Wright's generalized hypergeometric function

$${}_l\Psi_m[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m); z] =_l \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z]$$

is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \Sigma_{k=0}^{\infty}\{\prod_{t=0}^{l} \Gamma(\alpha_t + kA_t)\}\{\prod_{t=0}^{m} \Gamma(\beta_t + kB_t)\}^{-1} \frac{z^k}{k!}.$$

If $A_t = 1\,(t = 1, 2, \dots, l)$ and $B_t = 1\,(t = 1, 2, \dots, m)$ we have the relationship

$$\begin{aligned} \Omega_l \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] &\equiv_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) \\ &= \Sigma_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_l)_k}{(\beta_1)_k \dots (\beta_m)_k} \frac{z^k}{k!} \\ (l \leq m+1; l, m \in \mathbb{N}_0 &= \mathbb{N} = \{0, 1, 2, \dots, \};\ z \in \Delta). \end{aligned}$$

This is the generalized hypergeometric function (see [7]). Here (α_n) is the Pochammer symbol and $\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t)\right)$.

Using the generalized hypergeometric function,we define a linear operator

$$V[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \Sigma_P \to \Sigma_P$$

by

$$V[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = z^{-1}\Omega_l \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] * f(z). \quad (3)$$

For convenience, we denote $V[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]$ by $V[\alpha_1]$. If f has the form (1), then

$$V[\alpha_1]f(z) = \frac{1}{z} + \Sigma_{n=1}^{\infty} \sigma_n(\alpha_1) a_n z^n, \quad (4)$$

where

$$\sigma_n(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(n+1))\ldots\Gamma(\alpha_l + A_l(n+1))}{(k+1)!\Gamma(\beta_1 + B_1(n+1))\ldots\Gamma(\beta_l + B_l(n+1))}.$$

We now define a new subclass of Σ_P using the linear operator $V[\alpha_1]$.

Definition 1. *For $0 \leq \eta < 1$, $0 \leq \lambda < \frac{1}{2}$, $z \in \{z \in \mathbb{C}/0 < |z| < 1\}$ we say $f \in \Sigma_P$ is in $N_m^l(\lambda, \eta)$ if*

$$-Re\left(\frac{z(V[\alpha_1]f(z))' + \lambda z^2(V[\alpha_1]f(z))''}{(1-\lambda)(V[\alpha_1]f(z)) + \lambda z(V[\alpha_1]f(z))'}\right) > \eta.$$

Note that when $A_t = 1$ for all $t = 1, 2, \ldots, l$ and $B_t = 1$ for all $t = 1, 2, \ldots, m$, we get the class considered by Dziok et al. [5].

We now prove the coefficient inequality for $f \in N_m^l(\lambda, \eta)$.

Theorem 1. *Let $f \in \Sigma_P$ be given by (2). Then $f \in N_m^l(\lambda, \eta)$ if and only if*

$$\Sigma_{n=1}^{\infty}[(n+\eta)(n\lambda - \lambda + 1)]\sigma_n(\alpha_1)a_n \leq (1-\eta)(1-2\lambda). \qquad (5)$$

Proof. Since $f \in \Sigma_P$ given by (2) is in the class $N_m^l(\lambda, \eta)$,

$$-Re\left(\frac{z(V[\alpha_1]f(z))' + \lambda z^2(V[\alpha_1]f(z))"}{(1-\lambda)(V[\alpha_1]f(z)) + \lambda z(V[\alpha_1]f(z))'}\right) > \eta.$$

Substituting the series expansion for f we have

$$Re\left(\frac{\frac{-1}{z} + \Sigma_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n z^n + \frac{2\lambda}{z} + \Sigma_{n=1}^{\infty}\lambda n(n-1)\sigma_n(\alpha_1)a_n z^n}{(1-\lambda)(\frac{1}{z} + \Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n z^n) + \lambda(\frac{-1}{z} + \Sigma_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n z^n)}\right) \geq \eta.$$

That is,

$$Re\left(\frac{1 - \Sigma_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n z^{n+1} - 2\lambda - \Sigma_{n=1}^{\infty}\lambda n(n-1)\sigma_n(\alpha_1)a_n z^{n+1}}{(1-\lambda)(1 + \Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n z^{n+1}) + \lambda(-1 + \Sigma_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n z^{n+1})}\right) \geq \eta.$$

Allowing z to take only real values and as $z \to 1$, we get (5). Conversely, let $f \in \Sigma_P$ be given by (2) such that (5) holds. Let

$$w = \frac{-(z(V[\alpha_1]f(z))' + \lambda z^2(V[\alpha_1]f(z))")}{(1-\lambda)(V[\alpha_1]f(z)) + \lambda z(V[\alpha_1]f(z))'}.$$

We have to prove that $Rew > \eta$. It is enough to prove that

$$|w-1| < |w+1-2\eta|$$

$$\left|\frac{w-1}{w+1-2\eta}\right| = \left|\frac{-z(V[\alpha_1](f(z)))'-\lambda z^2(V[\alpha_1)f(z))''-(1-\lambda)(V[\alpha_1)f(z))-\lambda z(V[\alpha]f(z))'}{-z(V[\alpha_1]f(z))'-\lambda z^2(V[\alpha_1]f(z))''+(1-2\eta)(1-\lambda)(V[\alpha_1]f(z)+\lambda(1-2\eta)z(V[\alpha]f(z))'}\right|$$

$$= \left|\frac{-\Sigma_{n=1}^{\infty}[n+\lambda n(n-1)+\lambda n]\sigma_n(\alpha_1)a_n z^{n+1}}{2(1-\eta)(1-2\lambda)-\Sigma_{n=1}^{\infty}[n+\lambda n^2-\lambda n-1+\lambda+2\eta-2\eta\lambda]\sigma_n(\alpha_1)a_n z^{n+1}}\right|$$

$$< \frac{\Sigma_{n=1}^{\infty}[\lambda n^2+n+1-\lambda]\sigma_n(\alpha_1)a_n r^{n+1}}{2(1-\eta)(1-2\lambda)-\Sigma_{n=1}^{\infty}[\lambda n^2-2\lambda n+n-1+\lambda+2\eta-2\eta\lambda+2\eta\lambda n]\sigma_n(\alpha_1)a_n r^{n+1}}$$

$$< 1,$$

since the difference between denominator and numerator of the last expression equals $2[(1-\eta)(1-2\lambda)-\Sigma_{n=1}^{\infty}[\lambda n^2+n-\lambda n+\eta-\eta\lambda+n\eta\lambda]]$ which is non-negative, by (5).
This completes the proof.

From (5) we have

$$(1+\eta)\sigma_1 a_1 \leq \frac{(1-\eta)(1-2\lambda)}{1+\eta}. \tag{6}$$

Hence we may take

$$(1+\eta)\sigma_1 a_1 = \frac{(1-\eta)(1-2\lambda)c}{1+\eta}, \quad 0 < c < 1. \tag{7}$$

Following the works of Aouf and Darwish [1], Ghanim and Darus [7, 8], Magesh et al. [11] and Sivasubramanian et al. [13], we now introduce a class of functions and obtain the results analogous to the above mentioned works.

Definition 2. *The subclass* $N_m^l(\lambda,\eta,c)$ *of* $N_m^l(\lambda,\eta)$ *consists of all functions of the form*

$$f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta} z + \Sigma_{n=2}^{\infty}\sigma_n(\alpha_1)a_n z^n, \quad 0 < c < 1. \tag{8}$$

We now obtain the coefficient estimates, growth and distortion bounds, extreme points,radii of meromorphically starlikenss and convexity for the class $N_m^l(\lambda,\eta)$ by fixing the second coefficient.

2. Coefficient Inequality

We now prove the coefficient inequality.

Theorem 2. *Let f be defined by* (8). *Then $f \in N_m^l(\lambda, \eta, c)$ if and only if*

$$\Sigma_{n=2}^{\infty}[(n+\eta)(n\lambda-\lambda+1)]\sigma_n(\alpha_1)a_n \leq (1-\eta)(1-2\lambda)(1-c). \tag{9}$$

The result is sharp.

Proof. $f \in N_m^l(\lambda, \eta, c)$ implies $f \in N_m^l(\lambda, \eta)$. Therefore by (5)

$$(1+\eta)\sigma_1(\alpha_1)a_1 + \Sigma_{n=2}^{\infty}[(n+\eta)(n\lambda-\lambda+1)]\sigma_n(\alpha_1)a_n \leq (1-\eta)(1-2\lambda).$$

Using (7)

$$(1-\eta)(1-2\lambda)c + \Sigma_{n=2}^{\infty}[(n+\eta)(n\lambda-\lambda+1)]\sigma_n(\alpha_1)a_n \leq (1-\eta)(1-2\lambda)$$

from which we obtain (9). The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda)-\lambda+1)\sigma_n(\alpha_1)}z^n, \;\; n \geq 2. \tag{10}$$

Corollary 3. *If f defined by (8) is in the class $N_m^l(\lambda, \eta, c)$ then*

$$a_n \leq \frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}, \;\; n \geq 2. \tag{11}$$

The result is sharp for the function given by (10).

3. Growth and Distortion Theorems

We next prove the growth theorem for the class $N_m^l(\lambda, \eta, c)$.

Theorem 4. *If f given by (8) is in the class $N_m^l(\lambda, \eta, c)$ then for* $0 < |z| = r < 1$

$$|f(z)| \geq \frac{1}{r} - \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r - \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r^2 \tag{12}$$

and

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r + \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r^2. \tag{13}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}z^2$.

Proof. Since $f \in N_m^l(\lambda, \eta, c)$ by Theorem 2

$$\sigma_n(\alpha_1)a_n \leq \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\eta)(n\lambda-\lambda+1)}. \tag{14}$$

For $0 < |z| = r < 1$,

$$\begin{aligned}|f(z)| &\leq \frac{1}{|z|} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}|z| + \Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n|z|^n \\ &\leq \frac{1}{r} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r + r^2\Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n \\ &\leq \frac{1}{r} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r + \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|f(z)| &\geq \frac{1}{|z|} - \frac{(1-\eta)(1-2\lambda)c}{1+\eta}|z| - \Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n|z|^n \\ &\geq \frac{1}{r} - \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r - r^2\Sigma_{n=1}^{\infty}\sigma_n(\alpha_1)a_n \\ &\geq \frac{1}{r} - \frac{(1-\eta)(1-2\lambda)c}{1+\eta}r - \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r^2.\end{aligned}$$

The distortion theorem for the class $N_m^l(\lambda, \eta, c)$ is as follows:

Theorem 5. *If f given by (8) is in the class $N_m^l(\lambda, \eta, c)$ then for $0 < |z| = r < 1$*

$$|f'(z)| \geq \frac{1}{r^2} - \frac{(1-\eta)(1-2\lambda)c}{1+\eta} - \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r \tag{15}$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta} + \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}r. \tag{16}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \frac{(1-\eta)(1-2\lambda)(1-c)}{(1+\lambda)(2+\eta)}z^2$.

4. Extreme Points

Theorem 6. *Let $f_1(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z$ and for $n \geq 2$,*

$$f_n(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}\frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}z^n.$$

Then $f \in N_m^l(\lambda,\eta,c)$ *if and only if it can be expressed as*

$$f(z) = \Sigma_{n=1}^{\infty}\mu_n f_n(z),\ \mu_n \geq 0,\ \Sigma_{n=1}^{\infty}\mu_n = 1.$$

Proof. Suppose $f(z) = \Sigma_{n=1}^{\infty}\mu_n f_n(z)$, $\mu_n \geq 0$, $\Sigma_{n=1}^{\infty}\mu_n = 1$. Then

$$f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}\frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}\mu_n z^n.$$

Now

$$\Sigma_{n=2}^{\infty}\frac{(1-\eta)(1-2\lambda)(1-c)\mu_n}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}\frac{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}{(1-\eta)(1-2\lambda)(1-c)} = \Sigma_{n=2}^{\infty}\mu_n = 1-\mu_1 \leq 1.$$

This implies $f \in N_m^l(\lambda,\eta,c)$. Conversely, let $f \in N_m^l(\lambda,\eta,c)$. Then

$$a_n \leq \frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}\ ,\quad n \geq 2.$$

Set $\mu_n = \frac{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}{(1-\eta)(1-2\lambda)(1-c)}a_n,\ n \geq 2$ and $\mu_1 = 1 - \Sigma_{n=2}^{\infty}\mu_n$. Then $f(z) = \Sigma_{n=1}^{\infty}\mu_n f_n(z)$.

Theorem 7. *The class* $N_m^l(\lambda,\eta,c)$ *is closed under convex combination.*

Proof. Let $f, g \in N_m^l(\lambda,\eta,c)$ such that

$$f(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}a_n z^n$$

and

$$g(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}b_n z^n.$$

For $0 \leq \mu \leq 1$, let

$$h(z) = \mu f(z) + (1-\mu)g(z).$$

Then

$$h(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}[a_n\mu + (1-\mu)b_n]z^n.$$

Therefore

$$\Sigma_{n=2}^{\infty}[(n+\eta)(n\lambda-\lambda+1)]\sigma_n(\alpha_1)[a_n\mu + (1-\mu)b_n] \leq (1-\eta)(1-2\lambda)(1-c).$$

This implies $h(z) = \mu f(z) + (1-\mu)g(z) \in N_m^l(\lambda,\eta,c)$. Hence $N_m^l(\lambda,\eta,c)$ is closed under convex combination.

5. Radii of Meromorphically starlikeness and convexity

Theorem 8. *Let $f \in N_m^l(\lambda,\eta,c)$. Then f is meromorphically starlike of order δ $(0 \le \delta < 1)$ in the disk $|z| < r_1(\lambda,\eta,c,\delta)$, where $r_1(\lambda,\eta,c,\delta)$ is the largest value for which*

$$\left(\frac{(3-\delta)(1-\eta)(1-2\lambda)c}{1+\eta}\right)r^2 + \left(\frac{(n+2-\delta)(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)}\right)r^{n+1} \le 1-\delta, \;\; n \ge 2. \tag{17}$$

Proof. It is enough to show that

$$\left|\frac{zf'(z)}{f(z)} + 1\right| \le 1-\delta \tag{18}$$

$$\left|\frac{zf'(z)}{f(z)} + 1\right| = \left|\frac{zf'(z)+f(z)}{f(z)}\right| = \left|\frac{\frac{2(1-\eta)(1-2\lambda)cz^2}{1+\eta} + \Sigma_{n=2}^{\infty}(n+1)\sigma_n(\alpha_1)a_n z^{n+1}}{1+\frac{(1-\eta)(1-2\lambda)c}{1+\eta}z + \Sigma_{n=2}^{\infty}\sigma_n(\alpha_1)a_n z^{n+1}}\right|$$

(18) is true if

$$\left|\frac{2(1-\eta)(1-2\lambda)c}{1+\eta}z^2 + \Sigma_{n=2}^{\infty}(n+1)\sigma_n(\alpha_1)a_n z^{n+1}\right| \le (1-\delta)\left|1+\frac{(1-\eta)(1-2\lambda)c}{1+\eta}z^2 + \Sigma_{n=2}^{\infty}\sigma_n(\alpha_1)a_n z^{n+1}\right|.$$

That is

$$\frac{(3-\delta)(1-\eta)(1-2\lambda)c}{1+\eta}r^2 + \Sigma_{n=2}^{\infty}(n+2-\delta)a_n r^{n+1} \le 1-\delta.$$

From Theorem 1 we may take

$$a_n = \frac{(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)\sigma_n(\alpha_1)}\mu_n, \;\; n \ge 2, \;\; \mu_n \ge 0, \;\; \Sigma_{n=2}^{\infty}\mu_n = 1.$$

For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which $\frac{(n+2-\delta)\sigma_n(\alpha_1)}{(n+\eta)(n\lambda-\lambda+1)}r^{n+1}$ is maximal. This implies

$$\Sigma_{n=2}^{\infty}(n+2-\delta)\sigma_n(\alpha_1)a_n r^{n+1} \le \frac{(n_0+2-\delta)(1-\eta)(1-2\lambda)(1-c)}{(n_0+\eta)(n_0\lambda-\lambda+1)}r^{n_0+1}.$$

Then f is starlike of order δ in $0 < |z| < r_1(\lambda,\eta,c,\delta)$ if

$$\frac{(3-\delta)(1-\eta)(1-2\lambda)c}{1+\eta}r^2 + \frac{(n_0+2-\delta)(1-\eta)(1-2\lambda)(1-c)}{(n_0+\eta)(n_0\lambda-\lambda+1)}r^{n_0+1} \le 1-\delta.$$

We have to find the value of $r_0 = r_0(\lambda,\eta,c,\delta)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\delta)(1-\eta)(1-2\lambda)c}{1+\eta}r^2 + \frac{(n_0+2-\delta)(1-\eta)(1-2\lambda)(1-c)}{(n_0+\eta)(n_0\lambda-\lambda+1)}r^{n_0+1} = 1-\delta. \tag{19}$$

It is the value for which $f(z)$ is starlike of order δ in $0 < |z| < r_0$.

We now state a result for radius of meromorphic convexity for the class $N_m^l(\lambda, \eta, c)$ for which the proof is similar to above.

Theorem 9. *Let $f \in N_m^l(\lambda, \eta, c)$. Then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_2(\lambda, \eta, c, \delta)$ where $r_2(\lambda, \eta, c, \delta)$ is the largest value for $n \geq 2$,*

$$\left(\frac{(3-\delta)(1-\eta)(1-2\lambda)c}{1+\eta}\right)r^2 + \left(\frac{n(n+2-\delta)(1-\eta)(1-2\lambda)(1-c)}{(n+\eta)(n\lambda-\lambda+1)}\right)r^{n+1} \leq 1-\delta. \tag{20}$$

Remark 1. *By specializing the parameters in the Fox-Wright's generalized hypergeometric functions we obtain the class of Dziok et al. [5]. The corresponding class of fixed second coefficient can be defined and results analogue to the above can be obtained.*

Acknowledgements. The authors would like thank the referee for his/her insightful suggestions.

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Thomas Rosy and S. Sunil Varma
Department of Mathematics
Madras Christian College
Tambaram, Chennai
Tamil Nadu - 600059.
email: *thomas.rosy@gmail.com, sunu_79@yahoo.com*