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# RECURRENT GENERALIZED $(\kappa, \mu)$ SPACE FORMS

C.R.Premalatha and H.G.Nagaraja

Abstract. In this paper we study generalized $(k, \mu)$ space forms by considering certain curvature conditions like generalized recurrent, ricci recurrent, generalized $\phi$ recurrent conditions. We found relations among associated functions $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$ in $\phi$-concorcular recurrent, quasi-conformally $\phi$-flat and quasi-conformally flat generalized $(k, \mu)$ space forms.

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## 1. Introduction

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1}$$

where $f_1, f_2, f_3$ are some differentiable functions on $M$ and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$
$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

for any vector fields $X$, $Y$, $Z$ on $M$.

In [7], the authors defined a generalized $(k, \mu)$ space form as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor can be written as

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{2}$$

where $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$ are differentiable functions on $M$ and $R_1$, $R_2$, $R_3$ are tensors defined above and

$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$$
$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,$$
$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,$$

for any vector fields $X$, $Y$, $Z$, where $2h = L_\xi\phi$ and $L$ is the usual Lie derivative.This manifold was denoted by $M(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)$.
Natural examples of generalized $(k, \mu)$ space forms are $(k, \mu)$ space forms and generalized Sasakian space forms. The authors in [1] proved that contact metric generalized $(k, \mu)$ space forms are generalized $(k, \mu)$ spaces and if dimension is greater than or equal to 5, then they are $(k, \mu)$ spaces with constant $\phi-$sectional curvature $2f_6 - 1$. They gave a method of constructing examples of generalized $(k, \mu)$ space forms and proved that generalized $(k, \mu)$ space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [3], it is proved that under $D_a-$homothetic deformation generalized $(k, \mu)$ space form structure is preserved for dimension 3, but not in general. In this paper, we study generalized $(k, \mu)$ space forms under the curvature conditions like generalized recurrent, ricci recurrent, generalized $\phi$-recurrent, flat and $\phi$-flat conditions. The paper is organised as follows. After preliminaries in section 2, we study generalized recurrent generalized $(k, \mu)$ space forms in section 3 and found the condition for $M(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)$ to be co-symplectic. In section 4 we study generelized $\phi-$recurrent generalized $(k, \mu)$-space forms and found relations among associated functions. In sections 5 and 6 we study concircular curvature tensor and quasi-conformal curvature of generalized $(k, \mu)$ space forms.

## 2. Preliminaries

In this section, some general definitions and basic formulas are presented which will be used later. A (2n+1)-dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold if it admits a tensor field $\phi$ of type (1,1), a vector field $\xi$, and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{4}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \phi X) = 0,\ g(X, \xi) = \eta(X). \tag{5}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \phi Y)$ is the fundamental 2-form of $M$.

It is well known that on a contact metric manifold $(M, \phi, \xi, \eta, g)$, the tensor $h$ is defined by $2h = L_\xi \phi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, \;\; h\phi = -\phi h, \;\; trh = 0, \;\; \eta \circ h = 0, \tag{6}$$

$$\nabla_X \xi = -\phi X - \phi h X, \;\; (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{7}$$

In a $(2n + 1)$-dimensional $(k, \mu)$-contact metric manifold, we have [6]

$$h^2 = (k - 1)\phi^2, \;\; k \leq 1, \tag{8}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{9}$$

$$\begin{aligned}(\nabla_X h)(Y) =& [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi h X) \\ &- \mu\eta(X)\phi h Y.\end{aligned} \tag{10}$$

**Definition 1.** *A contact metric manifold $M$ is said to be*
*(i) Einstein if $S(X, Y) = \lambda g(X, Y)$, where $\lambda$ is a constant and $S$ is the Ricci tensor,*
*(ii) $\eta$-Einstein if $S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y)$, where $\alpha$ and $\beta$ are smooth functions on $M$.*

In a $(2n + 1)$-dimensional generalized $(k, \mu)$ space form, the following relations hold.

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{11}$$

$$\begin{aligned}QX =& [2nf_1 + 3f_2 - f_3]X + [(2n - 1)f_4 - f_6]hX \\ &- [3f_2 + (2n - 1)f_3]\eta(X)\xi,\end{aligned} \tag{12}$$

$$\begin{aligned}S(X, Y) =& [2nf_1 + 3f_2 - f_3]g(X, Y) + [(2n - 1)f_4 - f_6]g(hX, Y) \\ &- [3f_2 + (2n - 1)f_3]\eta(X)\eta(Y),\end{aligned} \tag{13}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{14}$$

$$r = 2n[(2n + 1)f_1 + 3f_2 - 2f_3], \tag{15}$$

where $Q$ is the Ricci operator, $S$ is the Ricci tensor and $r$ is the scalar curvature of $M(f_1, ..., f_6)$.
The relation between the associated functions $f_i, i = 1, ..., 6$ of $M(f_1, ..., f_6)$ was recently discussed by Carriazo et al. [7].

## 3. Generalized recurrent generalized $(k,\mu)$ space forms

A generalized $(k,\mu)$ space form $M(f_1,...,f_6)$ is called generalized recurrent,[8] if its curvature tensor $R$ satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z], \tag{16}$$

where $A$ and $B$ are two $1-forms$ and $B$ is non-zero.

**Theorem 1.** *A generalized recurrent $M(f_1,...,f_6)$ is co-symplectic provided $f_1 \neq f_3$.*

*Proof.* Taking $Y = W = \xi$ in (1), we obtain

$$(\nabla_X R)(\xi,Z)\xi = A(X)R(\xi,Z)\xi + B(X)[\eta(Z)\xi - Z]. \tag{17}$$

By the definition of covariant derative, we have

$$(\nabla_X R)(\xi,Z)\xi = \nabla_X R(\xi,Z)\xi - R(\nabla_X\xi,Z)\xi - R(\xi,\nabla_X Z)\xi - R(\xi,Z)\nabla_X\xi. \tag{18}$$

Using (2), (7) and (10), we get

$$\begin{aligned}(\nabla_X R)(\xi,Z)\xi &= X(f_1-f_3)[\eta(Z)\xi - Z] - X(f_4-f_6)hZ - (f_4-f_6)[(1-k)g(X,\phi Z)\xi \\ &+ g(X,h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1-f_3)\nabla_X Z - (f_4-f_6)g(-\phi X - \phi hX, hZ)\xi.\end{aligned} \tag{19}$$

Now comparing (17) and (19), we obtain

$$\begin{aligned}&[(X - A(X))(f_1-f_3) - B(X)][\eta(Z)\xi - Z] \\ &+ [(A(X) - X)(f_4-f_6)]hZ - (f_4-f_6)[(1-k)g(X,\phi Z)\xi \\ &+ g(X,h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1-f_3)\nabla_X Z + (f_4-f_6)g(\phi X + \phi hX, hZ)\xi = 0.\end{aligned} \tag{20}$$

Taking $Z = \xi$ in (20), we obtain

$$(f_1-f_3)(\nabla_X\xi) = 0. \tag{21}$$

Thus $M$ is co-symplectic if $f_1 \neq f_3$. Hence the proof.

## Ricci-recurrent generalized $(k,\mu)$ space forms

A generalized $(k,\mu)$- space form $M(f_1,...,f_6)$ is generalized Ricci-recurrent [9], if its Ricci tensor $S$ satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + 2nB(X)g(Y,Z), \tag{22}$$

where $A$ and $B$ are two non-zero $1-$forms.

**Theorem 2.** *In a generalized Ricci-recurrent $M(f_1, ..., f_6)$, $f_1 \neq f_3$ holds. Further the $1-$forms $A(X)$ and $B(X)$ are related by* (28).

*Proof.* By the definition of covariant derivative, we have

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi). \tag{23}$$

Using (7) and (14) in (23), we get

$$(\nabla_X S)(Y, \xi) = 2nd(f_1 - f_3)(X)\eta(Y) + 2n(f_1 - f_3)g(X + hX, \phi Y) + (2nf_1 + 3f_2 - f_3)g(Y, \phi X + \phi hX) + [(2n-1)f_4 - f_6]g(hY, \phi X + \phi hX). \tag{24}$$

Taking $Z = \xi$ in (22) and using (5) and (14), we obtain

$$(\nabla_X S)(Y, \xi) = 2n(f_1 - f_3)A(X)\eta(Y) + 2nB(X)\eta(Y). \tag{25}$$

From (24) and (25), we obtain

$$2nd(f_1 - f_3)(X)\eta(Y) + 2n(f_1 - f_3)g(X + hX, \phi Y) + (2nf_1 + 3f_2 - f_3)g(Y, \phi X + \phi hX) + [(2n-1)f_4 - f_6]g(hY, \phi X + \phi hX) - 2n(f_1 - f_3)A(X)\eta(Y) - 2nB(X)\eta(Y) = 0. \tag{26}$$

Taking $Y = \xi$ in (26), we obtain

$$2nd(f_1 - f_3)(X) = 2n(f_1 - f_3)A(X) - 2nB(X). \tag{27}$$

If $(f_1 - f_3) = c$, a constant, then (27) reduces to

$$B(X) = cA(X). \tag{28}$$

Since $B(X)$ is not zero, we have $f_1 \neq f_3$.

It is easy to see that a generalized recurrent $M(f_1, ..., f_6)$ is always generalized Ricci-recurrent. It follows from theorem 1 and theorem 2 that

**Corollary 3.** *A generalized recurrent $M(f_1, ..., f_6)$ is always co-symplectic.*

## 4. Generalized $\phi-$recurrent $M(f_1, ..., f_6)$

A generalized $(k, \mu)$ space form $M(f_1, ..., f_6)$ is called

**Definition 2.** *A generalized $\phi-$Ricci recurrent [4],[9], if*

$$\phi^2((\nabla_X Q)(Y)) = A(X)QY + 2nB(X)Y \tag{29}$$

**Definition 3.** *$\phi-$Ricci symmetric, if*

$$\phi^2((\nabla_X Q)(Y)) = 0, \tag{30}$$

*where $Q$ is the Ricci operator, $A(X)$ and $B(X)$ are non-zero $1-$forms.*

**Theorem 4.** *In a generalized $(k, \mu)$ space form which is $\phi-$Ricci recurrent the relation $3f_2 + (2n-1)f_3 = 0$ holds.*

*Proof.* Using (4) and (3), we have

$$-\nabla_X QY + Q(\nabla_X Y) + \eta((\nabla_X Q)Y)\xi = A(X)QY + 2nB(X)Y. \tag{31}$$

Taking $Y = \xi$ in (31) and contracting with respect to $Z$, we obtain

$$\begin{aligned} &- g(\nabla_X Q\xi, Z) + g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)\xi)\eta(Z) \\ &= A(X)g(Q\xi, Z) + 2nB(X)\eta(Z) \end{aligned} \tag{32}$$

Using (7) and (12) in (32), we obtain

$$\begin{aligned} &2n(f_1 - f_3)[g(\phi X, Z) + g(\phi hX, Z)] - S(\phi X, Z) - S(\phi hX, Z) \\ &= 2n[(f_1 - f_3)A(X) + B(X)]\eta(Z). \end{aligned} \tag{33}$$

Replacing $X$ by $\phi X$ in (33), we get

$$\begin{aligned} &2n(f_1 - f_3)[g(\phi^2 X, Z) + g(\phi h\phi X, Z)] - S(\phi^2 X, Z) - S(\phi h\phi X, Z) \\ &= 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \end{aligned} \tag{34}$$

Using (3), (13) and (14) in (34), we get

$$\begin{aligned} S(X, Z) =& [2n(f_1 - f_3) - ((2n-1)f_4 - f_6)(k-1)]g(X, Z) + [3f_2 + (2n-1)f_3]g(hX, Z) \\ &+ (k-1)[(2n-1)f_4 - f_6]\eta(X)\eta(Z) + 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \end{aligned} \tag{35}$$

Replacing $Z$ by $\phi Z$ in (35), we get

$$S(X, \phi Z) = [2n(f_1 - f_3) - ((2n-1)f_4 - f_6)(k-1)]g(X, \phi Z) + [3f_2 + (2n-1)f_3]g(hX, \phi Z). \tag{36}$$

Again from (13), we have

$$S(X, \phi Z) = [2nf_1 + 3f_2 - f_3)]g(X, \phi Z) + [(2n-1)f_4 - f_6]g(hX, \phi Z). \tag{37}$$

From (37) and (36), we get

$$3f_2 + (2n-1)f_3 = 0. \tag{38}$$

If $A(X)$ and $B(X)$ are zero in (35), then $M(f_1, ..., f_6)$ is called $\phi-$Ricci symmetric [9].
It is easy to see that relation (38) holds in $\phi-$Ricci symmetric generalized $(k, \mu)$ space form .
Conversely suppose $3f_2 + (2n-1)f_3 = 0$ holds in $\phi-$Ricci symmetric generalized $(k, \mu)$ space form, then from (12)

$$QY = (2nf_1 + 3f_2 - f_3)Y + [(2n-1)f_4 - f_6]hY.$$

Differentiating covariantly with respect to $X$, we obtain

$$(\nabla_X Q)Y = \nabla_X((2nf_1 + 3f_2 - f_3)Y) + \nabla_X(((2n-1)f_4 - f_6)hY).$$

Applying $\phi^2$ on both sides, we obtain

$$\phi^2((\nabla_X Q)Y) = d(2nf_1 + 3f_2 - f_3)(X)\phi^2 Y + d((2n-1)f_4 - f_6)(X)\phi h^2 Y.$$

Therefore $M(f_1, ..., f_6)$ is $\phi-$Ricci symmetric if and only if $2nf_1 + 3f_2 - f_3$ and $(2n-1)f_4 - f_6$ are constants.

## 5. Concircular curvature tensor of generalized $(k, \mu)$ space forms

The Concircular curvature tensor of $M(f_1, ..., f_6)$ is given by [11]

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y]. \tag{39}$$

$M(f_1, ..., f_6)$ is said to be

**Definition 4.** *$\phi-$concircular recurrent[12], if*

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)\tilde{C}(X, Y)Z, \tag{40}$$

*where $A(W)$ is a non-zero $1-$form.*

**Definition 5.** *:$\phi-$concircular symmetric, if*

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = 0. \tag{41}$$

**Theorem 5.** *In a $\phi-$concircular recurrent generalized $(k, \mu)$ space form, the relation $(2n - 1)f_3 + 3f_2 = 0$ holds.*

*Proof.* Taking the covariant differentiation of (5), we get

$$(\nabla_W \tilde{C})(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{dr(W)}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]. \tag{42}$$

Applying $\phi^2$ on both sides, we get

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z) - \frac{dr(W)}{2n(2n + 1)}[g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y]. \tag{43}$$

Suppose $M(f_1, ..., f_6)$ is $\phi-$concircular recurrent. Then from (3) and (40) in (43) and taking $X = \xi$, we obtain

$$A(W)\tilde{C}(\xi, Y)Z = -(\nabla_W R)(\xi, Y)Z + \eta((\nabla_W R)(\xi, Y)Z)\xi + \frac{dr(W)}{2n(2n + 1)}\eta(Z)\phi^2 Y. \tag{44}$$

Suppose the vector fields $X$, $Y$ and $Z$ are orthogonal to $\xi$. Then taking $X = \xi$ in (5) and using (2), and (3), we get

$$\tilde{C}(\xi, Y)Z = [(f_1 - f_3) - \frac{r}{2n(2n + 1)}]g(Y, Z)\xi + (f_4 - f_6)g(hZ, Y)\xi. \tag{45}$$

Now using (2), (3) and (45) in (44) and contracting with respect to $\xi$, we obtain

$$A(W)\left[\left((f_1 - f_3) - \frac{r}{2n(2n + 1)}\right)g(Y, Z) + (f_4 - f_6)g(hZ, Y)\right] = 0. \tag{46}$$

Taking $Z = \xi$ in (46) and using (15), we obtain

$$(2n - 1)f_3 + 3f_2 = 0. \tag{47}$$

### 5.1. Concircular curvature tensor of $(k, \mu)$ space forms

In a $(k, \mu)$ space form $M$, curvature tensor $R$ is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{c+3}{4})[g(Y,Z)X - g(X,Z)Y] \\ &+ (\frac{c-1}{4})[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ &+ (\frac{c+3}{4} - k)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ &+ [g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] \\ &+ \frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX] \\ &+ (1-\mu)[\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi], \end{aligned} \tag{48}$$

where $c$ is the constant $\phi-$sectional curvature of $M$.
From (48), we have

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(Y,hZ)\xi - \eta(Z)hY], \tag{49}$$

$$r = n[c(n+1) + 3(n-1) + 4k]. \tag{50}$$

**Theorem 6.** *In a $\phi-$concircular recurrent $(k, \mu)$ space form, the constant $\phi-$sectional curvature of $M$ is given by $c = \frac{k(4n-2)-3(n-1)}{n+1}$.*

*Proof.* Suppose $M$ is $\phi-$concircular recurrent. Then from (3) and (40) in (43) and taking $X = \xi$, we obtain

$$A(W)\tilde{C}(\xi,Y)Z = -(\nabla_W R)(\xi,Y)Z + \eta((\nabla_W R)(\xi,Y)Z)\xi + \frac{dr(W)}{2n(2n+1)}\eta(Z)\phi^2 Y. \tag{51}$$

Suppose the vector fields $X$, $Y$ and $Z$ are orthogonal to $\xi$. Then taking $X = \xi$ in (5) and using (49), (50) and (3), we get

$$\tilde{C}(\xi,Y)Z = \left(k - \frac{c(n+1) + 3(n-1) + 4k}{2(2n+1)}\right) g(Y,Z)\xi + \mu g(hZ,Y)\xi. \tag{52}$$

Using (52), (49) and (3) in (51) and contracting with respect to $\xi$, we obtain

$$A(W)\left[\left(k - \frac{c(n+1) + 3(n-1) + 4k}{2(2n+1)}\right) g(Y,Z) + \mu g(hZ,Y)\right] = 0. \tag{53}$$

Taking $Z = \xi$ in (53), we get

$$c = \frac{k(4n-2) - 3(n-1)}{n+1}. \tag{54}$$

## 6. Quasi-conformal curvature tensor on generalized $(k, \mu)$ space forms

In a $(2n+1)$-dimensional generalized $(k, \mu)$ space form $M(f_1, ..., f_6)$, the quasi-conformal curvature tensor [11] is given by

$$W(X,Y)Z = aR(X,Y)Z + b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] - \frac{a+2b(2n)}{2n(2n+1)}r[g(Y,Z)X - g(X,Z)Y], \tag{55}$$

where $a$ and $b$ are arbitrary constants such that $ab \neq 0$.

**Definition 6.** *A generalized $(k, \mu)$ space form $M(f_1, ..., f_6)$ is said to be quasi-conformally $\phi-$flat if*

$$g(W(X,Y)Z, \phi W) = 0. \tag{56}$$

**Definition 7.** *A generalized $(k, \mu)$ space form $M(f_1, ..., f_6)$ is said to be quasi-conformally flat if*

$$W(X,Y)Z = 0. \tag{57}$$

### 6.1. Quasi-conformally $\phi-$flat generalized $(k, \mu)$ space forms

In a (2n+1)-dimensional almost contact metric manifold $M$, [10], if $\{e_1, ...e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, ...\phi e_{2n}, \xi\}$ is also a local orthonormal basis and

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{58}$$

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y,Z) - S(Y,\xi)\eta(Z), \tag{59}$$

$$\sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \tag{60}$$

for all $Y$, $Z \in TM$. In a generalized $(k, \mu)$ space form, we have

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r - 2n(f_1 - f_3), \tag{61}$$

$$\sum_{i=1}^{2n} R(e_i, \acute{Y}, Z, e_i) = \sum_{i=1}^{2n} R(\phi e_i, \acute{Y}, Z, \phi e_i)$$
$$= S(Y, Z) - ((f_1 - f_3)[g(Y, Z) - \eta(Z)\eta(Y)] + (f_4 - f_6)g(hZ, Y)) . \tag{62}$$

**Theorem 7.** *A quasi-conformally $\phi$-flat generalized $(k, \mu)$ space form is an $\eta-$Einstein manifold.*

*Proof.* From (55), we have

$$\begin{aligned} & g(W(X, Y)Z, \phi W) = \\ & ag(R(X, Y)Z, \phi W) + b[S(X, Z)g(Y, \phi W) - S(Y, Z)g(X, \phi W) + g(X, Z)S(Y, \phi W) \\ & - g(Y, Z)S(X, \phi W)] - \frac{a + 2b(2n)}{2n(2n+1)} r[g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)], \end{aligned} \tag{63}$$

for $X$, $Y$, $Z$, $W \in TM$.
For a local orthonormal basis $\{e_1, ...e_{2n}, \xi\}$ of vector fields in $M(f_1, ..., f_6)$, putting $X = \phi e_i$ and $W = e_i$ in (63) and using (58), (59), (60), (61) and (62), we obtain

$$\begin{aligned} \sum_{i=1}^{2n} g(W(\phi e_i, Y)Z, \phi e_i) &= a[S(Y, Z) - (f_1 - f_3)(g(Y, Z) - \eta(Z)\eta(Y)) \\ & - (f_4 - f_6)g(hZ, Y)] + b[(2 - 2n)S(Y, Z) - S(\xi, Z)\eta(Y) - S(Y, \xi)\eta(Z) \\ & - g(Y, Z)(r - 2n(f_1 - f_3))] - \frac{a + 2b(2n)}{2n(2n+1)} r[g(Y, Z)2n - g(\phi Y, \phi Z)]. \end{aligned} \tag{64}$$

If $M(f_1, ..., f_6)$ is quasiconformally $\phi$-flat, then (64) reduces to

$$[b(2n - 2) - a]S(Y, Z) = pg(Y, Z) + q\eta(Y)\eta(Z) - a(f_4 - f_6)g(hZ, Y), \tag{65}$$

where

$$\begin{aligned} p &= -a(f_1 - f_3) - b(r - 2n(f_1 - f_3)) - m(2n - 1), \\ q &= a(f_1 - f_3) - 4nb(f_1 - f_3) - m, \\ m &= \frac{a + 2b(2n)}{2n(2n+1)} r. \end{aligned}$$

Replacing $Z$ by $hZ$ in (65) and using (13) and (8), we obtain

$$g(hZ, Y) = \frac{[-a(f_4 - f_6) - (b(2n - 2) - a)e]}{[b(2n - 2) - a]t - p} (k - 1)[\eta(Z)\eta(Y) - g(Y, Z)], \tag{66}$$

with

$$t = 2nf_1 + 3f_2 - f_3,$$
$$e = (2n-1)f_4 - f_6.$$

Now substituting for $g(hZ,Y)$ in (65), we obtain

$$S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z), \tag{67}$$

where

$$\alpha = \frac{p+l}{b(2n-2)-a},$$
$$\beta = \frac{q-l}{b(2n-2)-a},$$
$$l = \frac{a(f_4-f_6)[-a(f_4-f_6)-(b(2n-2)-a)e]}{(b(2n-2)-a)t-p}(k-1).$$

Therefore $M(f_1,...,f_6)$ is an $\eta-$Einstein.

Putting $Z=\xi$ in (65) and using (14), we obtain

$$2nb(f_1-f_3)(2n-1) - 2na(f_1-f_3) = -br - \left(\frac{a+4nb}{2n+1}r\right). \tag{68}$$

If $f_1 = f_3$, then from (68), we have

$$r = 0 \ or \ a+b+6nb = 0.$$

Thus we have

**Proposition 8.** *In a quasi-conformally $\phi$-flat $M(f_1,...,f_6)$, either $r=0$ or $a+b+6nb=0$ provided $f_1=f_3$.*

## 6.2. Quasi-conformally flat generalized $(k,\mu)$ space forms

**Theorem 9.** *In a quasi-conformally flat generalized $(k,\mu)$ space form which is $\phi-$ricci recurrent the scalar curvature is given by $\frac{-(2bt+m)}{a}$.*

*Proof.* Suppose $M(f_1,...,f_6)$ is Quasi-conformally flat, then from (55) and (57), we obtain

$$\begin{aligned} aR(X,Y)Z = &- b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] \\ &+ \frac{a+2b(2n)}{2n(2n+1)}r[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{69}$$

Using (12) and (13) in the above equation, it reduces to

$$\begin{aligned} R(X,Y)Z =& \frac{1}{a}(-(2bt+m)[g(X,Z)Y - g(Y,Z)X] - be[g(hX,Z)Y - g(hY,Z)X] \\ &+bs[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &-be[g(X,Z)hY - g(Y,Z)hX] + bs[\eta(X)Y - \eta(Y)X]\eta(Z))\,, \end{aligned} \tag{70}$$

where

$$\begin{aligned} t &= 2nf_1 + 3f_2 - f_3, \\ s &= 3f_2 + (2n-1)f_3, \\ e &= (2n-1)f_4 - f_6, \\ m &= \frac{a+2b(2n)}{2n(2n+1)}r. \end{aligned}$$

If $M(f_1, ..., f_6)$ is $\phi-$Ricci recurrent, then $s = 0$ and $e = 0$.
Then from (70), we obtain

$$R(X,Y)Z = (\frac{-(2bt+m)}{a})[g(X,Z)Y - g(Y,Z)X]. \tag{71}$$

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C.R.Premalatha and H.G.Nagaraja*
Department of Mathematics, Bangalore University, Central College Campus,
Bengaluru – 560 001, INDIA
email: *premalathacr@yahoo.co.in, hgnraj@yahoo.com*
*Corresponding author.