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A STUDY ON CURVATURE TENSOR OF A GENERALIZED SASAKIAN SPACE FORM

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Abstract. In this paper we study some results of C-Bochner curvature tensor and τ-curvature tensor of a generalized Sasakian space form.

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1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold and further it is known that the sectional curvature of a manifold determines curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real space form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{1}$$

A Sasakian manifold (M, ϕ, ξ, η, g) is said to be a Sasakian space form if all the ϕ-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant c, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. In such a case, Riemannian curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2}$$

In 2004, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the concept of generalized Sasakian space forms. The generalized Sasakian space form is defined as follows:

A generalized Sasakian space form is an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{3}$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. Sasakian-space-forms appear as natural examples of generalized Sasakian space forms, with constant functions $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The generalized Sasakian space forms have been extensively studied by [2, 3, 4, 7, 18, 20].

A Riemannian manifold is called locally symmetric if its curvature tensor R is parallel, that is $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold, the notion of semi-symmetric manifold was defined by

$$(R(X,Y) \cdot R)(U,V)W = 0. \tag{4}$$

In this paper, we study some results of C-Bochner curvature tensor and τ-curvature tensor in generalized Sasakian space forms.

2. Preliminaries

In this section, we give some general definitions and basic formulas which we will use later:
A $(2n+1)$-dimensional Riemannian manifold M is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \tag{5}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{6}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{7}$$

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y), \tag{8}$$

where Φ is called the fundamental two-form of the manifold. If ξ is a Killing vector field, then the contact metric manifold is called a K-contact manifold. It is well known that a contact metric manifold is K-contact if and only if

$$\nabla_X \xi = -\phi X, \tag{9}$$

for any vector field X on M. An almost contact metric manifold is Sasakian if it is normal and it satisfies the condition,

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{10}$$

for any vector fields X and Y.
From equation (3), we have

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, & (11)\\ R(X,\xi)\xi &= (f_1 - f_3)\{X - \eta(X)\xi\}, & (12)\\ R(\xi,X)Y &= (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, & (13)\\ R(X,\xi)Y &= (f_1 - f_3)\{\eta(Y)X - g(X,Y)\xi\}. & (14)\end{aligned}$$

Again from (3) and by taking an account of $S(X,Y) = \sum_{i=1}^{(2n+1)} g(R(e_i,X)Y, e_i)$, we get

$$S(X,Y) = [2nf_1 + 3f_2 - f_3]g(X,Y) + [-3f_2 - (2n-1)f_3]\eta(X)\eta(Y). \tag{15}$$

From (15), we have

$$\begin{aligned} QX &= [2nf_1 + 3f_2 - f_3]X + [-3f_2 - (2n-1)f_3]\eta(X)\xi, & (16)\\ r &= 2n(2n+1)f_1 + 6nf_2 - 4nf_3. & (17)\end{aligned}$$

where Q is the Ricci operator and r is the scalar curvature of $M(f_1, f_2, f_3)$.
Putting $Y = \xi$ in (15), we get

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{18}$$

3. Flat C-Bochner curvature tensor in generalized Sasakian space forms

In 1969, Matsumoto and Chuman [13] introduced the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties. Later the properties of C-Bochner curvature tensor is extensively studied by many authors like H. Endo [9], M.M. Tripathi [12], C.S. Bagewadi [10], U.C. De [8], A. A. Shaikh [19], etc.

The C-Bochner curvature tensor B [12] is given by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{2n+4}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX + S(X,Z)Y \\ &+ g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X + S(\phi X,Z)\phi Y \\ &+ 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi \\ &+ \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] - \frac{D+2n}{2n+4}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ 2g(\phi X,Y)\phi Z] + \frac{D}{2n+4}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ &- \eta(X)g(Y,Z)\xi] - \frac{D-4}{2n+4}[g(X,Z)Y - g(Y,Z)X]. \qquad (19) \end{aligned}$$

where $D = \frac{r+2n}{2n+2}$ and R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature of the manifold respectively.

Theorem 1. *If C-Bochner curvature tensor is zero in generalized Sasakian space form then it is an η-Einstein manifold and the scalar curvature r is given by*

$$r = 2n(2nf_1 + 3f_2 - f_3) + \frac{2n}{3}(f_1 - f_3) + \frac{4n}{3}. \qquad (20)$$

Proof. We assume that $B(X,Y)Z = 0$. Then from (19), we have

$$\begin{aligned} R(X,Y,Z,W) &= -\frac{1}{2n+4}[g(X,Z)S(Y,W) - S(Y,Z)g(X,W) - g(Y,Z)S(X,W) \\ &+ S(X,Z)g(Y,W) + g(\phi X,Z)S(\phi Y,W) - S(\phi Y,Z)g(\phi X,W) \\ &- g(\phi Y,Z)S(\phi X,W) + S(\phi X,Z)g(\phi Y,W) + 2S(\phi X,Y)g(\phi Z,W) \\ &+ 2g(\phi X,Y)S(\phi Z,W) + \eta(Y)\eta(Z)S(X,W) - \eta(Y)\eta(W)S(X,Z) \\ &+ \eta(X)\eta(W)S(Y,Z) - \eta(X)\eta(Z)S(Y,W)] \\ &+ \frac{r+2n(2n+3)}{(2n+2)(2n+4)}[g(\phi X,Z)g(\phi Y,Z) - g(\phi Y,Z)g(\phi X,W) \\ &+ 2g(\phi X,Y)g(\phi Z,W)] - \frac{r+2n}{(2n+2)(2n+4)}[g(X,Z)\eta(Y)\eta(W) \\ &- \eta(Y)\eta(Z)g(X,W) + \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z)] \\ &+ \frac{r+2n-4(2n+2)}{(2n+2)(2n+4)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \qquad (21) \end{aligned}$$

Putting $X = W = e_i$, where $\{e_i : i = 1, 2, \ldots, (2n+1)\}$ is a local orthonormal basis of vector fields in generalized Sasakian space form $M(f_1, f_2, f_3)$ and by virtue of

$S(X,Y) = \sum_{i=1}^{(2n+1)} g(R(e_i, X, Y), e_i)$, we get

$$\begin{aligned} S(Y,Z) &= [2nf_1 + 3f_2 - f_3]g(Y,Z) \\ &+ \left[-(2nf_1 + f_2 - f_3) + \frac{2n}{3}(f_1 - f_3) + \frac{4n}{3}\right]\eta(Y)\eta(Z). \end{aligned} \tag{22}$$

Therefore, the manifold is η-Einstein.
On contracting (22), we obtain (20). This completes the proof of the theorem.

4. Generalized Sasakian space form satisfying $R(X,Y) \cdot B = 0$

Theorem 2. *If in a generalized Sasakian space form of dimension* $(2n+1)$*, the relation* $R(X,Y) \cdot B = 0$ *holds with the condition* $f_1 \neq f_3$*, then the manifold is an* η*-Einstein and the scalar curvature is* r *is given by*

$$r = 2n(2nf_1 + 3f_2 - f_3) + \frac{2n(4n+3)}{3}(f_1 - f_3) - \frac{n(8n+11)}{3}. \tag{23}$$

Proof. Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form. From equation (19), we obtain

$$\begin{aligned} \eta(B(X,Y)Z) &= \eta(R(X,Y)Z) + \frac{1}{2n+4}[g(X,Z)S(Y,\xi) - S(Y,Z)\eta(X) - g(Y,Z)S(X,\xi) \\ &+ S(X,Z)\eta(Y) + \eta(Y)\eta(Z)S(X,\xi) - \eta(Y)S(X,Z) + \eta(X)S(Y,Z) \\ &- \eta(X)\eta(Z)S(Y,\xi)] + \frac{4}{2n+4}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \end{aligned} \tag{24}$$

Putting $X = \xi$ in (24) and by virtue of (3), (15) and (13), we get

$$\eta(B(\xi,Y)Z) = \left[(f_1 - f_3) - \frac{2n(f_1 - f_3)}{2n+4} - \frac{4}{2n+4}\right][g(Y,Z) - \eta(Y)\eta(Z)]. \tag{25}$$

On taking $Z = \xi$ in (24) and by virtue of (11) and (15), we get

$$\eta(B(X,Y)\xi) = 0. \tag{26}$$

Now, we define

$$\begin{aligned} (R(X,Y) \cdot B)(U,V)Z &= R(X,Y)B(U,V)Z - B(R(X,Y)U,V)Z \\ &- B(U,R(X,Y)V)Z - B(U,V)R(X,Y)Z. \end{aligned} \tag{27}$$

We assume that $R(X,Y) \cdot B = 0$. Then we have

$$\begin{aligned} &R(X,Y)B(U,V)Z - B(R(X,Y)U,V)Z \\ &-B(U,R(X,Y)V)Z - B(U,V)R(X,Y)Z = 0, \end{aligned} \tag{28}$$

which implies that

$$(f_1 - f_3)[B^{'}(U,V,Z,Y) - \eta(Y)\eta(B(U,V)Z) - g(U,Y)\eta(B(\xi,V)Z) + \eta(U)\eta(B(Y,V)Z) - g(Y,V)\eta(B(U,\xi)Z) + \eta(V)\eta(B(U,Y)Z) - g(Y,Z)\eta(B(U,V)\xi) + \eta(Z)\eta(B(U,V)Y)] = 0, \tag{29}$$

where $B^{'}(U,V,Z,Y) = g(B(U,V)Z,Y)$.

The above equation (29) states that either

$$f_1 - f_3 = 0$$

or

$$[B^{'}(U,V,Z,Y) - \eta(Y)\eta(B(U,V)Z) - g(U,Y)\eta(B(\xi,V)Z) + \eta(U)\eta(B(Y,V)Z) - g(Y,V)\eta(B(U,\xi)Z) + \eta(V)\eta(B(U,Y)Z) - g(Y,Z)\eta(B(U,V)\xi) + \eta(Z)\eta(B(U,V)Y)] = 0. \tag{30}$$

If $f_1 \neq f_3$ then equation (30) must be true. Now, we proceed under the assumption that $f_1 \neq f_3$. Putting $U = Y = e_i$ in (30), where $\{e_i : i = 1, 2, \ldots, (2n+1)\}$ is a local orthonormal basis of vector fields, we have

$$\sum_{i=1}^{(2n+1)} B^{'}(e_i,V,Z,e_i) - \sum_{i=1}^{(2n+1)} \eta(e_i)\eta(B(e_i,V)Z) - \sum_{i=1}^{(2n+1)} g(e_i,e_i)\eta(B(\xi,V)Z) + \sum_{i=1}^{(2n+1)} \eta(e_i)\eta(B(e_i,V)Z) - \sum_{i=1}^{(2n+1)} g(e_i,V)\eta(B(e_i,\xi)Z) + \sum_{i=1}^{(2n+1)} \eta(V)\eta(B(e_i,e_i)Z) - \sum_{i=1}^{(2n+1)} g(e_i,Z)\eta(B(e_i,V)\xi) + \sum_{i=1}^{(2n+1)} \eta(Z)\eta(B(e_i,V)e_i) = 0. \tag{31}$$

By using (24), (25) and (26) in (31), we have

$$\begin{aligned} S(V,Z) &= \left[(2nf_1 + 3f_2 - f_3) + \frac{4n}{3}(f_1 - f_3) - \frac{4n}{3}\right] g(V,Z) \\ &+ \left[-(2nf_1 + 3f_2 - f_3) + \frac{2n}{3}(f_1 - f_3) - \frac{7n}{3}\right] \eta(V)\eta(Z). \end{aligned} \tag{32}$$

And by contracting (32), we have (23). This completes the proof of the theorem.

5. Generalized Sasakian space form satisfying $R(\xi, X) \cdot \tau = 0$

M.M. Tripathi and et. al. ([15, 16]) introduced the τ-tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like M-projective curvature tensor, W_i-curvature tensor ($i = 0, \dots, 9$) and W_j^*-curvature tensor ($j = 0, 1$). M.M. Tripathi and et. al. studied τ-curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. H.G. Nagaraja et. al. [17] studied the τ-curvature tensor in (k, μ)-contact manifolds. In this section, we study the generalized Sasakian space form satisfying $R(\xi, X) \cdot \tau = 0$, where τ is a τ-curvature tensor and is given by

$$\begin{aligned} \tau(X, Y)Z &= a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y + a_3 S(X, Y)Z \\ &+ a_4 g(Y, Z)QX + a_5 g(X, Z)QY + a_6 g(X, Y)QZ \\ &+ a_7[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{33}$$

where $a_0, \dots, a_7$ are some smooth functions on M.
By taking an innerproduct with respect to ξ in (33), we have

$$\begin{aligned} \eta(\tau(X, Y)Z) &= a_0(f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + a_1 S(Y, Z)\eta(X) \\ &+ a_2 S(X, Z)\eta(Y) + a_3 S(X, Y)\eta(Z) + 2n(f_1 - f_3)a_4 g(Y, Z)\eta(X) \\ &+ 2n(f_1 - f_3)a_5 g(X, Z)\eta(Y) + 2n(f_1 - f_3)a_6 g(X, Y)\eta(Z) \\ &+ a_7[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{34}$$

Theorem 3. *If in a generalized Sasakian space form of dimension* $(2n + 1)$, *the relation* $R(\xi, X) \cdot \tau = 0$ *holds with the condition* $f_1 \neq f_3$, *then the manifold is an* η*-Einstein provided* $a_0 + a_5 + a_6 \neq 0$.

Proof. We assume $R(\xi, X) \cdot \tau = 0$, then we have

$$\begin{aligned} & R(\xi, X)\tau(Y, Z)W - \tau(R(\xi, X)Y, Z)W \\ & -\tau(Y, R(\xi, X)Z)W - \tau(Y, Z)R(\xi, X)W = 0. \end{aligned} \tag{35}$$

By using (13) in (35), we obtain

$$\begin{aligned} & (f_1 - f_3)[g(X, \tau(Y, Z)W)\xi - \eta(\tau(Y, Z)W)X - g(X, Y)\tau(\xi, Z)W \\ & +\eta(Y)\tau(X, Z)W - g(X, Z)\tau(Y, \xi)W + \eta(Z)\tau(Y, X)W - g(X, W)\tau(Y, Z)\xi \\ & +\eta(W)\tau(Y, Z)X] = 0. \end{aligned} \tag{36}$$

By taking an innerproduct with respect to ξ in (36), we have

$$\begin{aligned} & (f_1 - f_3)[g(X, \tau(Y, Z)W) - \eta(\tau(Y, Z)W)\eta(X) - g(X, Y)\eta(\tau(\xi, Z)W) \\ & +\eta(Y)\eta(\tau(X, Z)W) - g(X, Z)\eta(\tau(Y, \xi)W) + \eta(Z)\eta(\tau(Y, X)W) \\ & -g(X, W)\eta(\tau(Y, Z)\xi) + \eta(W)\eta(\tau(Y, Z)X)] = 0, \end{aligned} \tag{37}$$

from (37), either $(f_1 - f_3) = 0$ or

$$[g(X, \tau(Y, Z)W) - \eta(\tau(Y, Z)W)\eta(X) - g(X, Y)\eta(\tau(\xi, Z)W) + \eta(Y)\eta(\tau(X, Z)W) - g(X, Z)\eta(\tau(Y, \xi)W) + \eta(Z)\eta(\tau(Y, X)W) - g(X, W)\eta(\tau(Y, Z)\xi) + \eta(W)\eta(\tau(Y, Z)X)] = 0. \tag{38}$$

If $f_1 \neq f_3$ then equation (38) must be true. Now, we proceed under the assumption that $f_1 \neq f_3$. By using (33), (34) in (38) and on simplification, we get

$$\begin{aligned}
&a_0 g(X, R(Y, Z)W) + a_4 g(Z, W)S(X, Y) + a_5 g(Y, W)S(Z, X) \\
&+a_6 g(Y, Z)S(W, X) + (f_1 - f_3)a_0[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] \\
&-2n(f_1 - f_3)a_2 g(X, Y)\eta(Z)\eta(W) - 2n(f_1 - f_3)a_3 g(X, Y)\eta(Z)\eta(W) \\
&-2n(f_1 - f_3)a_4 g(X, Y)g(Z, W) + a_2 S(X, W)\eta(Y)\eta(Z) + a_3 S(X, Z)\eta(Y)\eta(W) \\
&-2n(f_1 - f_3)a_1 g(X, Z)\eta(Y)\eta(W) - 2n(f_1 - f_3)a_3 g(X, Z)\eta(Y)\eta(W) \\
&-2n(f_1 - f_3)a_5 g(X, Z)g(Y, W) + a_1 S(X, W)\eta(Y)\eta(Z) + a_3 S(Y, X)\eta(W)\eta(Z) \\
&-2n(f_1 - f_3)a_1 g(X, W)\eta(Y)\eta(Z) - 2n(f_1 - f_3)a_2 g(X, W)\eta(Y)\eta(Z) \\
&+a_1 S(Z, X)\eta(Y)\eta(W) - 2n(f_1 - f_3)a_6 g(Y, Z)g(X, W) \\
&+a_2 S(Y, X)\eta(Z)\eta(W) = 0.
\end{aligned} \tag{39}$$

Putting $X = Y = e_i$, in (39), where $\{e_i : i = 1, 2, \dots, (2n+1)\}$ is a local orthonormal basis of vector fields and on simplification, we have

$$\begin{aligned}
S(Z, W) &= \left[\frac{2n(f_1 - f_3)(a_0 + a_5 + a_6) + [2n(2n+1)(f_1 - f_3) - r]a_4}{(a_0 + a_5 + a_6)}\right] g(Z, W) \\
&+ \left[\frac{[2n(2n+1)(f_1 - f_3) - r](a_2 + a_3)}{(a_0 + a_5 + a_6)}\right] \eta(Z)\eta(W).
\end{aligned} \tag{40}$$

This completes the proof of the above theorem.

6. Generalized Sasakian space form satisfying $\tau(\xi, X) \cdot S = 0$

Theorem 4. *If in a generalized Sasakian space form of dimension $(2n + 1)$, the relation $\tau(\xi, X) \cdot S = 0$ holds, then the manifold is an η-Einstein provided* $[(f_1 - f_3)[2n(a_1 + a_2) - a_0] + a_5(2nf_1 + 3f_2 - f_3) - a_7 r] \neq 0$.

Proof. In a generalized Sasakian space form the following condition satisfies:

$$(\tau(\xi, X) \cdot S)(Y, Z) = 0, \tag{41}$$

that is

$$S(\tau(\xi, X)Y, Z) + S(Y, \tau(\xi, X)Z) = 0. \tag{42}$$

By using (33) in (42) and by virtue of (15), (16) and (18), we obtain

$$\begin{aligned}
&(f_1 - f_3)a_0[2n(f_1 - f_3)g(X, Y)\eta(Z) - S(X, Z)\eta(Y)] \\
&+2na_1(f_1 - f_3)\eta(Z)S(X, Y) + 2na_2(f_1 - f_3)\eta(Y)S(X, Z) \\
&+2na_3(f_1 - f_3)\eta(X)S(Y, Z) + 4n^2a_4(f_1 - f_3)^2g(X, Y)\eta(Z) \\
&+a_5[2nf_1 + 3f_2 - f_3]\eta(Y)S(X, Z) + a_5[-3f_2 - (2n - 1)f_3]\eta(Y)\eta(X)S(\xi, Z) \\
&+a_6[2nf_1 + 3f_2 - f_3]\eta(X)S(Y, Z) + a_6[-3f_2 - (2n - 1)f_3]\eta(Y)\eta(X)S(\xi, Z) \\
&+a_7r[2n(f_1 - f_3)\eta(Z)g(X, Y) - \eta(Y)S(X, Z)] + a_0(f_1 - f_3)[2n(f_1 - f_3)\eta(Y)g(X, Z) \\
&-\eta(Z)S(X, Y)] + 2na_1(f_1 - f_3)\eta(Y)S(X, Z) + 2na_2(f_1 - f_3)\eta(Z)S(Y, X) \\
&+2na_3(f_1 - f_3)\eta(X)S(Y, Z) + 4n^2a_4(f_1 - f_3)^2g(X, Z)\eta(Y) \\
&+a_5(2nf_1 + 3f_2 - f_3)\eta(Z)S(X, Y) + a_5[-3f_2 - (2n - 1)f_3]\eta(Z)\eta(X)S(\xi, Y) \\
&+a_6[2nf_1 + 3f_2 - f_3]\eta(X)S(Y, Z) + a_6[-3f_2 - (2n - 1)f_3]\eta(Z)\eta(X)S(\xi, Y) \\
&+a_7r[2n(f_1 - f_3)\eta(Y)g(X, Z) - \eta(Z)S(X, Y)] = 0.
\end{aligned} \tag{43}$$

Putting $Z = \xi$ in (43), we have

$$\begin{aligned}
S(X, Y) = &\frac{-[2n(f_1 - f_3)[(f_1 - f_3)(a_0 + 2na_4) + a_7r)]]}{[(f_1 - f_3)[2n(a_1 + a_2) - a_0] + a_5(2nf_1 + 3f_2 - f_3) - a_7r]}g(X, Y) \\
&-\frac{2n(f_1 - f_3)[2n(f_1 - f_3)[a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6] + a_5[-3f_2 - (2n - 1)f_3]]}{[(f_1 - f_3)[2n(a_1 + a_2) - a_0] + a_5(2nf_1 + 3f_2 - f_3) - a_7r]} \\
&\times\eta(X)\eta(Y).
\end{aligned} \tag{44}$$

This completes the proof of the theorem.

7. C-Bochner Recurrent in generalized Sasakian space form

A generalized Sasakian space form is said to be C-Bochner recurrent in generalized Sasakian space form if it satisfies

$$(\nabla_W B)(X, Y)Z = A(W)B(X, Y)Z, \tag{45}$$

where A is a non-zero 1-form and B is a C-Bochner curvature tensor. We define a function $f^2 = g(B, B)$ on M, where the metric g is extended to the inner product between the tensor fields. Then, we know that

$$f(Yf) = f^2A(Y). \tag{46}$$

This implies that

$$Yf = fA(Y), \qquad (f \neq 0). \tag{47}$$

From above equation, we have

$$X(Yf) - Y(Xf) = f\{XA(Y) - YA(X) - A([X,Y])\}. \tag{48}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$, then we have

$$dA(X,Y) = 0, \tag{49}$$

that is 1-form A is closed.

Now, from $(\nabla_V B)(X,Y)Z = A(V)B(X,Y)Z$, we have

$$(\nabla_U \nabla_V B)(X,Y)Z = \{UA(V) + A(U)A(V)\}B(X,Y)Z. \tag{50}$$

By the above equation, we have

$$(R(X,Y) \cdot B)(U,V)Z = [2dA(X,Y)]B(U,V)Z. \tag{51}$$

By using (49) in (51), we have

$$(R(X,Y) \cdot B)(U,V)Z = 0. \tag{52}$$

Hence by virtue of Theorem 4, we can state the following:

Theorem 5. *If C-Bochner curvature tensor is recurrent in generalized Sasakian space form then it is an η-Einstein manifold.*

Corollary 6. *In a C-Bochner recurrent generalized Sasakian space form the 1-form, A is closed.*

8. Ricci semi-symmetric generalized Sasakian space form

A generalized Sasakian space form is said to be Ricci semi-symmetric if it satisfies

$$(R(X,Y) \cdot S)(U,V) = 0, \tag{53}$$

that is

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{54}$$

Putting $Y = V = \xi$ in the above equation, we have

$$S(R(X,\xi)U,\xi) + S(U,R(X,\xi)\xi) = 0. \tag{55}$$

By using (12) and (14) in (55), we have

$$\begin{aligned}&(f_1 - f_3)\{\eta(U)S(X,\xi) - g(X,U)S(\xi,\xi)\}\\ &+(f_1 - f_3)[S(X,U) - \eta(X)S(U,\xi)] = 0.\end{aligned} \tag{56}$$

Again by using (15) in (56), we have

$$S(X,U) = 2n(f_1 - f_3)g(X,U). \tag{57}$$

Hence, we state the following:

Theorem 7. *A Ricci semi-symmetric generalized Sasakian space form is an Einstein manifold.*

Corollary 8. *A Ricci semi-symmetric generalized Sasakian space form is an Einstein manifold with $f_1 \neq f_3$. Otherwise, that is if $f_1 = f_3$ then it is Ricci flat [$S(X,U) = 0$].*

9. Generalized Sasakian space form satisfying $S(\xi,X) \cdot R = 0$

Theorem 9. *If in a generalized Sasakian space form of dimension $(2n+1)$, the relation $S(X,Y) \cdot R = 0$ holds, then the manifold is an η-Einstein manifold.*

Proof. Using the following equations

$$\begin{aligned}S((X,\xi)\cdot R)(U,V)W &= ((X \wedge_S \xi)\cdot R)(U,V)W, \\ &= (X \wedge_S \xi)R(U,V)W + R((X \wedge_S \xi)U,V)W \\ &\quad +R(U,(X \wedge_S \xi)V)W + R(U,V)(X \wedge_S \xi)W,\end{aligned} \tag{58}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \tag{59}$$

we have

$$\begin{aligned}S((X,\xi)\cdot R)(U,V)W &= S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi + S(\xi,U)R(X,V)W \\ &\quad -S(X,U)R(\xi,V)W + S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W \\ &\quad +S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi.\end{aligned} \tag{60}$$

By using the condition $S(\xi, X) \cdot R = 0$, we get

$$\begin{aligned} & S(\xi, R(U,V)W)X - S(X, R(U,V)W)\xi + S(\xi, U)R(X,V)W \\ & -S(X,U)R(\xi,V)W + S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W \\ & +S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi = 0. \end{aligned} \tag{61}$$

By taking an inner product with respect to ξ in the above equation and by virtue of (18), we have

$$\begin{aligned} & 2n(f_1 - f_3)\eta(R(U,V)W)\eta(X) - S(X, R(U,V)W) + 2n(f_1 - f_3)\eta(U)\eta(R(X,V)W) \\ & -S(X,U)\eta(R(\xi,V)W) + 2n(f_1 - f_3)\eta(V)R(U,X)W - S(X,V)\eta(R(U,\xi)W) \\ & +2n(f_1 - f_3)\eta(W)\eta(R(U,V)X) - S(X,W)\eta(R(U,V)\xi) = 0, \end{aligned} \tag{62}$$

putting $U = W = \xi$ and by virtue of (12), (13) and (14), we have

$$S(X,V) = -2n(f_1 - f_3)g(X,V) + 4n(f_1 - f_3)\eta(X)\eta(V). \tag{63}$$

This completes the proof.

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