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INCLUSION AND NEIGHBORHOOD PROPERTIES OF CERTAIN CLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS DEFINED BY USING A DIFFERENTIAL OPERATOR

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Abstract. Making use of a differential operator, we introduce and investigate two classes of multivalently analytic functions of complex order. In this paper, we obtain coefficient estimates and inclusion relationships involving the (j,δ)-neighborhood of various subclasses of multivalently analytic functions of complex order.

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1. Introduction

Let $T(j,p)$ denote the class of functions of the form:

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k \quad (a_k\geq 0; p,j\in N=\{1,2,....\}), \tag{1.1}$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$.

A function $f(z)\in T(j,p)$ is said to be p-valently starlike of order α if it satisfies the inequality:

$$Re\left\{\frac{zf^{'}(z)}{f(z)}\right\}>\alpha \quad (z\in U;\ \ 0\leq\alpha<p; p\in N). \tag{1.2}$$

We denote by $T_j^*(p,\alpha)$ the class of all p-valently starlike functions of order α.

Also a function $f(z)\in T(j,p)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\}>\alpha \quad (z\in U; 0\leq\alpha<p; p\in N). \tag{1.3}$$

We denote by $C_j(p,\alpha)$ the class of all p-valently convex functions of order α.

We note that (see for example Duren [10] and Goodman [12])

$$f(z)\in C_j(p,\alpha)\Longleftrightarrow \frac{zf'(z)}{p}\in T_j^*(p,\alpha)\quad (0\le\alpha<p;p\in N). \tag{1.4}$$

For each $f(z)\in T(j,p)$, we have (see [9])

$$f^{(q)}(z)=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}a_kz^{k-q}\quad (q\in N_0=N\cup\{0\};p>q). \tag{1.5}$$

For a function $f(z)$ in $T(j,p)$, we define

$$D_p^0f^{(q)}(z)=f^{(q)}(z)\ ,$$

$$D_p^1f^{(q)}(z)=Df^{(q)}(z)=\frac{z}{(p-q)}(f^{(q)}(z))'=\frac{z}{(p-q)}f^{(1+q)}(z)$$

$$=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)a_kz^{k-q}, \tag{1.6}$$

$$D_p^2f^{(q)}(z)=D(D_p^1f^{(q)}(z))$$

$$=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^2a_kz^{k-q}, \tag{1.7}$$

and

$$D_p^nf^{(q)}(z)=D(D_p^{n-1}f^{(q)}(z))\qquad (n\in N)$$

$$=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^na_kz^{k-q}$$

$$(p,j\in N;q\in N_0;p>q\,). \tag{1.8}$$

The differential operator $D_p^nf^{(q)}(z)$ was introduced by Aouf [6, 7]. We note that, when $q=0$ and $p=1$, the differential operator $D_1^n=D^n$ was introduced by Salagean [19]. Also when $q=0$, the operator D_p^n was introduced by Kamali and Orhan [13], Aouf [5] and Aouf and Mostafa [8].

Now, making use of the differential operator $D_p^nf^{(q)}(z)$ given by (1.8), we introduce a new subclass $R_j(n,p,q,b,\beta)$ of the p-valently analytic function class $T(j,p)$ satisfying the following inquality:

$$\left| \frac{1}{b} \left(\frac{z(D_p^n f^{(q)}(z))'}{D_p^n f^{(q)}(z)} - (p-q) \right) \right| < \beta$$

$$(z \in U; p, j \in N; q, n \in N_0; b \in C \backslash \{0\}; 0 < \beta \leq 1; p > q). \tag{1.9}$$

Now, following the earlier investigations by Goodman [11], Ruscheweyh [18], and others including Altintas and Owa [1], Altintas et al. ([2] and [3]), Murugusundaramoorthy and Srivastava [14], Raina and Srivastava [17], Aouf [4] and Srivastava and Orhan [20] (see also [15], [16] and [21]), we define the (j, δ)-neighborhood of a function $f(z) \in T(j, p)$ by (see, for example, [3, p. 1668])

$$N_{j,\delta}(f) = \left\{ g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \text{ and } \sum_{k=j+p}^{\infty} k \left| a_k - b_k \right| \leq \delta \right\}. \tag{1.10}$$

In particular, if

$$h(z) = z^p \ (p \in N), \tag{1.11}$$

then we immediately have

$$N_{j,\delta}(h) = \left\{ g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \text{ and } \sum_{k=j+p}^{\infty} k \left| b_k \right| \leq \delta \right\}. \tag{1.12}$$

Also, let $L_j(n, p, q, b, \beta)$ denote the subclass of $T(j, p)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left| \frac{1}{b} \left(\frac{(D_p^n f^{(q)}(z))'}{(p-q) z^{p-q-1}} - \theta(p,q) \right) \right| < \beta$$

$$(z \in U; p, j \in N; q, n \in N_0; b \in C \backslash \{0\}; 0 < \beta \leq 1; p > q), \tag{1.13}$$

where

$$\theta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} 1 & (q = 0), \\ p(p-1)...(p-q+1) & (q \neq 0). \end{cases} \tag{1.14}$$

Remark 1. *Throughout our present paper, we assume that* $\theta(p, q)$ *is defined by* (1.14).

2. Neighborhoods for the classes $R_j(n,p,q,b,\beta)$ and $L_j(n,p,q,b,\beta)$

In our investigation of the inclusion relations involving $N_{j,\delta}(h)$, we shall require Lemmas 1 and 2 below.

Lemma 1. *Let the function $f(z) \in T(j,p)$be defined by* (1.1)*. Then $f(z)$ is in the class $R_j(n,p,q,b,\beta)$ if and only if*

$$\sum_{k=j+p}^{\infty} \left(\frac{k-q}{p-q}\right)^n (k+\beta\,|b|-p)\theta(k,q)a_k \leq \beta\,|b|\,\theta(p,q)\ . \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $R_j(n,p,q,b,\beta)$. Then, in view of (1.8) and (1.9), we obtain the following inequality:

$$Re\left\{\frac{z(D_p^n f^{(q)}(z))'}{D_p^n f^{(q)}(z)} - (p-q)\right\} > -\beta\,|b| \quad (z \in U), \tag{2.2}$$

or, equivalently,

$$Re\left\{\frac{-\sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n (k-p)\theta(k,q)a_k z^{k-p}}{\theta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \theta(k,q)a_k z^{k-p}}\right\} > -\beta\,|b| \quad (z \in U). \tag{2.3}$$

Setting $z = r$ $(0 \leq r < 1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for all r $(0 < r < 1)$. Thus, by letting $r \to 1^-$ through real values, (2.3) leads us to the desired assertion (2.1) of Lemma 1.

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find from (1.9) that

$$\left|\frac{z(D_p^n f^{(q)}(z))'}{D_p^n f^{(q)}(z)} - (p-q)\right| = \left|\frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n (k-p)\theta(k,q)a_k z^{k-p}}{\theta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \theta(k,q)a_k z^{k-p}}\right|$$

$$\leq \frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n (k-p)\theta(k,q)a_k}{\theta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \theta(k,q)a_k} \leq \frac{\beta\,|b|\left\{\theta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \theta(k,q)a_k\right\}}{\theta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \theta(k,q)a_k} = \beta\,|b|\,.$$

Hence, by the maximum modulus theorem, we have $f(z) \in R_j(n,p,q,b,\beta)$, which evidenlty completes the proof of Lemma 1.

Similarly, we can prove the following lemma.

Lemma 2. *Let the function* $f(z) \in T(j,p)$ *be defined by* (1.1). *Then* $f(z) \in L_j(n,p,q,b,\beta)$ *if and only if*

$$\sum_{k=j+p}^{\infty} \left(\frac{k-q}{p-q}\right)^{n+1} \theta(k,q) a_k \leq \beta\,|b|\,. \tag{2.4}$$

Our first inclusion relation involving $N_{j,\delta}(h)$ is given in the following theorem.

Theorem 3. *Let*

$$\delta = \frac{(j+p)\beta\,|b|\,\theta(p,q)}{\left(\frac{j+p-q}{p-q}\right)^{n} (j+\beta\,|b|)\theta(j+p,q)} \quad (p > |b|), \tag{2.5}$$

then

$$R_j(n,p,q,b,\beta) \subset N_{j,\delta}(h). \tag{2.6}$$

Proof. Let $f(z) \in R_j(n,p,q,b,\beta)$. Then, in view of the assertion (2.1) of Lemma 1, we have

$$\left(\frac{j+p-q}{p-q}\right)^{n} (j+\beta\,|b|)\theta(j+p,q) \sum_{k=j+p}^{\infty} a_k$$

$$\leq \sum_{k=j+p}^{\infty} \left(\frac{k-q}{p-q}\right)^{n} (k+\beta\,|b|-p)\theta(k,q) a_k \leq \beta\,|b|\,\theta(p,q), \tag{2.7}$$

which readily yields

$$\sum_{k=j+p}^{\infty} a_k \leq \frac{\beta\,|b|\,\theta(p,q)}{\left(\frac{j+p-q}{p-q}\right)^{n} (j+\beta\,|b|)\theta(j+p,q)}. \tag{2.8}$$

Making use of (2.1) again, in conjunction with (2.8), we get

$$\left(\frac{j+p-q}{p-q}\right)^{n} \theta(j+p,q) \sum_{k=j+p}^{\infty} k a_k$$

$$\leq \beta\,|b|\,\theta(p,q) + (p-\beta\,|b|) \left(\frac{j+p-q}{p-q}\right)^{n} \theta(j+p,q) \sum_{k=j+p}^{\infty} a_k$$

$$\leq \beta\,|b|\,\theta(p,q) + \frac{\beta\,|b|\,(p-\beta\,|b|)\theta(p,q)}{(j+\beta\,|b|)} = \frac{(j+p)\beta\,|b|\,\theta(p,q)}{(j+\beta\,|b|)}.$$

Hence

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(j+p)\beta\,|b|\,\theta(p,q)}{\left(\frac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q)} = \delta \quad (p > |b|) \tag{2.9}$$

which, by means of the definition (1.12), establishes the inclusion relation (2.6) asserted by Theorem 3.

In a similar manner, by applying the assertion (2.4) of Lemma 2 instead of the assertion (2.1) of Lemma 1 to functions in the class $L_j(n,p,q,b,\beta)$, we can prove the following inclusion relationship.

Theorem 4. *If*

$$\delta = \frac{(j+p)\beta\,|b|}{\left(\dfrac{j+p-q}{p-q}\right)^{n+1} \theta(j+p,q)}\ , \tag{2.10}$$

then

$$L_j(n,p,q,b,\beta) \subset N_{j,\delta}(h). \tag{2.11}$$

3. Neighborhoods for the classes $R_j^{(\alpha)}(n,p,q,b,\beta)$ and $L_j^{(\alpha)}(n,p,q,b,\beta)$

In this section, we determine the neighborhood for each of the classes

$$R_j^{(\alpha)}(n,p,q,b,\beta) \text{ and } L_j^{(\alpha)}(n,p,q,b,\beta),$$

which we define as follows.

A function $f(z) \in T(j,p)$ is said to be in the class $R_j^{(\alpha)}(n,p,q,b,\beta)$ if there exists a function $g(z) \in R_j(n,p,q,b,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \quad (z \in U;\ 0 \leq \alpha < p). \tag{3.1}$$

Analogously, a function $f(z) \in T(j,p)$ is said to be in the class $L_j^{(\alpha)}(n,p,q,b,\beta)$ if there exists a function $g(z) \in L_j(n,p,q,b,\beta)$ such that the inequality (3.1) holds true.

Theorem 5. *If* $g(z) \in R_j(n,p,q,b,\beta)$ *and*

$$\alpha = p - \frac{\delta\left(\dfrac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q)}{(j+p)\left\{\left(\dfrac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q) - \beta\,|b|\,\theta(p,q)\right\}}, \tag{3.2}$$

then

$$N_{j,\delta}(g) \subset R_j^{(\alpha)}(n,p,q,b,\beta), \tag{3.3}$$

where

$$\delta \leq p(j+p)\left\{1-\beta\,|b|\,\theta(p,q)\left[\left(\frac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q)\right]^{-1}\right\}. \tag{3.4}$$

Proof. Suppose that $f(z) \in N_{j,\delta}(g)$. We find from (1.10) that

$$\sum_{k=j+p}^{\infty} k\,|a_k - b_k| \leq \delta, \tag{3.5}$$

which readily implies that

$$\sum_{k=j+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{j+p} \quad (p,j \in N). \tag{3.6}$$

Next, since $g(z) \in R_j(n,p,q,b,\beta)$, we have [cf. equation (2.8)]

$$\sum_{k=j+p}^{\infty} b_k \leq \frac{\beta\,|b|\,\theta(p,q)}{\left(\frac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q)}, \tag{3.7}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum\limits_{k=j+p}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=j+p}^{\infty} b_k}$$

$$\leq \frac{\delta}{j+p}.\frac{\left(\frac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q)}{\left\{\left(\frac{j+p-q}{p-q}\right)^n (j+\beta\,|b|)\theta(j+p,q) - \beta\,|b|\,\theta(p,q)\right\}} = p-\alpha, \tag{3.8}$$

provided that α is given by (3.2). Thus, by the above definition, $f(z) \in R_j^{(\alpha)}(n,p,q,b,\beta)$ for α given by (3.2). This evidently proves Theorem 5.

The proof of Theorem 6 below is similar to that of Theorem 5.

Theorem 6. *If $g(z) \in L_j(n,p,q,b,\beta)$ and*

$$\alpha = p - \frac{\delta \left(\dfrac{j+p-q}{p-q}\right)^{n+1} \theta(j+p,q)}{(j+p)\left\{\left(\dfrac{j+p-q}{p-q}\right)^{n+1} \theta(j+p,q) - \beta\,|b|\right\}}, \tag{3.9}$$

then

$$N_{j,\delta}(g) \subset L_j^{(\alpha)}(n,p,q,b,\beta), \tag{3.10}$$

where

$$\delta \leq p(j+p)\left\{1 - \beta\,|b|\left[\left(\frac{j+p-q}{p-q}\right)^{n+1} \theta(j+p,q)\right]^{-1}\right\}. \tag{3.11}$$

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