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A CLASS OF NEW DIFFERENCE SEQUENCE SPACES AND THEIR MATRIX TRANSFORMATIONS

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Abstract. In this paper, we define a class of new difference sequence spaces $\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$, where $\Delta^\nu_{[k]}x_k = k\nu_k x_k - (k+1)\nu_{k+1}x_{k+1}$ for all $k = 1, 2, 3...$ and $\nu = (\nu_k)$ is a fixed sequence of non zero complex numbers satisfying some conditions. Subsequently, we also derive some inclusion relations and topological properties of these spaces and discuss about their $p\alpha-, p\beta-$, and $p\gamma-$ duals. Finally, we introduce the concept of statistical convergence on these spaces and their matrix transformations.

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1. Introduction and preliminaries

Let ω be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the set of linear spaces that are bounded, convergent and null sequences $x = (x_k)$ with the complex terms respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N} = \{1, 2, 3...\}$, the set of positive integers. The notion of difference sequence space was introduced by Kızmaz [1] by defining the sequence space

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}, \tag{1}$$

for $X = \ell_\infty, c$ and c_0, where $\Delta x = (x_k - x_{k+1})$. Later on the above idea was generalized by Et and Çolok [2]. Subsequently, this concept was further extended and studied by Et and Esi [3], Et and Nuray [4], Baliarsingh[5], Et and Basarır[6] and many others (see [7]-[13]).

Let $\nu = (\nu_k)$ be any fixed sequence of non zero complex numbers satisfying

$$\lim_{k\to\infty} \inf_k |\nu_k|^{\frac{1}{k}} = r, \quad (0 < r \leq \infty).$$

Now, we define

$$\begin{aligned}
\ell_\infty(\Delta^\nu_{[k]}) &= \Big\{x = (x_k) \in \omega : \sup_k |\Delta^\nu_{[k]} x_k| < \infty\Big\},\\
c_0(\Delta^\nu_{[k]}) &= \Big\{x = (x_k) \in \omega : \lim_{k\to\infty} |\Delta^\nu_{[k]} x_k| = 0\Big\},\\
c(\Delta^\nu_{[k]}) &= \Big\{x = (x_k) \in \omega : \lim_{k\to\infty} |\Delta^\nu_{[k]} x_k - L| = 0, \ \text{ for some } L\Big\},
\end{aligned}$$

where $\Delta^\nu_{[k]} x_k = k\nu_k x_k - (k+1)\nu_{k+1} x_{k+1}$, for all $k \in \mathbb{N}$.
In particular, we have the following observations:

(i) For $\Delta^\nu_{[k]} = I$, an identity operator these classes generalize the basic sequence spaces ℓ_∞, c and c_0.

(ii) For $\nu_k = 1, \ (k \in \mathbb{N})$, these classes reduce to the sets of spaces $\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$, where $\Delta_{[k]} x_k = kx_k - (k+1)x_{k+1}$ (see [12]).

2. Topological properties and inclusion relations

In this section, we establish some new relations and basic topological properties concerning the spaces $\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]}), c_0(\Delta^\nu_{[k]}), \ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$.

Theorem 1. *$\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$ are linear over $\mathbb{C}$, the field of complex scalars under co-ordinate wise addition and scalar multiplication.*

Proof. The proof is a routine verification, hence omitted.

Corollary 2. *$\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$ are linear over $\mathbb{C}$, the field of complex scalars under co-ordinate wise addition and scalar multiplication.*

Theorem 3. *$\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$ are normed linear over with the norm*

$$\| x \|_{\Delta^\nu_{[k]}} = |\nu_1 x_1| + \sup_k |\Delta^\nu_{[k]} x_k|. \tag{2}$$

Proof. The proof is a routine verification, hence omitted.

Corollary 4. *$\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$ are normed linear spaces with the norm*

$$\| x \|_{\Delta_{[k]}} = |x_1| + \sup_k |\Delta_{[k]} x_k|. \quad (3)$$

Theorem 5. *$\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$ are compete normed linear spaces with the norm defined in (2).*

Proof. We give the proof for the space $\ell_\infty(\Delta^\nu_{[k]})$ only and for other spaces it follows the similar techniques. Suppose $x^M = (x^M_k), x^N = (x^N_k)$ are two elements of $\ell_\infty(\Delta^\nu_{[k]})$ for every $k, M, N \in \mathbb{N}$ and

$$\| x^M - x^N \|_{\Delta^\nu_{[k]}} = |\nu_1(x^N_1 - x^M_1)| + \sup_k |\Delta^\nu_{[k]}(x^N_k - x^M_k)| \to 0, \quad \text{as } M, N \to \infty$$

For every $\epsilon > 0$, there exists a number N_0 such that $M, N > N_0$

$$|\nu_1(x^N_1 - x^M_1)| < \epsilon \quad \text{and} \quad \sup_k |\Delta^\nu_{[k]}(x^N_k - x^M_k)| < \epsilon,$$

$\Rightarrow$ $(\nu_1 x^N_1)$ is a Cauchy sequence in $\mathbb{C}$. Again for $M, N > N_0$ and $k > 1$

$$\sup_k |\Delta^\nu_{[k]} x^N_k - \Delta^\nu_{[k]} x^M_k| < \epsilon$$

$\Rightarrow$ $|k\nu_k x^N_k - (k+1)\nu_{k+1} x^N_{k+1} - k\nu_k x^M_k + (k+1)\nu_{k+1} x^M_{k+1}| < \epsilon$

$\Rightarrow$ By putting $k = 1$, subsequently we get $(\nu_2 x^N_2)$ is a Cauchy sequence in $\mathbb{C}$.

Continuing this process one can show that $(\nu_k x^N_k)$ and $(\Delta^\nu_{[k]} x^N_k)$ are Cauchy sequences in $\mathbb{C}$ for all k and for $k > 1$ respectively. By complement of $\mathbb{C}$, $\lim_{N\to\infty} \Delta^\nu_{[k]} x^N_k = x_k$ for each fixed $k > 1$.
For given $\epsilon > 0$ and $N_0 > M, N$

$$\lim_{M\to\infty} \sup_k |\Delta^\nu_{[k]}(x^N_k - x^M_k)| = \sup_k |\Delta^\nu_{[k]} x^N_k - x_k| < \epsilon \quad (4)$$

Now

$$\lim_{M\to\infty} \| x^N - x^M \|_{\Delta^\nu_{[k]}} = |\nu_1(x^N_1 - x_1)| + \sup_k |\Delta^\nu_{[k]} x^N_k - x_k| \le 2\epsilon.$$

$\Rightarrow$ $x^N \to x$ as $M \to \infty$.

Since $\ell_\infty(\Delta^\nu_{[k]})$ is linear and $x = x - x^N + x^N$, this implies $x \in \ell_\infty(\Delta^\nu_{[k]})$. This completes the proof.

Corollary 6. *$\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$ are compete normed linear spaces with the norm defined in (3).*

Theorem 7. *$\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$ are BK-spaces under the norm defined in (2).*

Proof. We give the proof of the space $\ell_\infty(\Delta^\nu_{[k]})$ only and for the other spaces it will follow the similar technique. Let us consider the mapping

$$T : \ell_\infty(\Delta^\nu_{[k]}) \to \ell_\infty(\Delta^\nu_{[k]})$$

defined by $Tx = y = (0, x_2, x_3, ...)$, where $x = (x_k) = (x_1, x_2, x_3...)$. It is clear that T is bounded linear operator on $\ell_\infty(\Delta^\nu_{[k]})$.
The space $T(\ell_\infty(\Delta^\nu_{[k]})) = \{x = (x_k) : x_1 = 0, x \in \ell_\infty(\Delta^\nu_{[k]})\}$ is a subspace of $\ell_\infty(\Delta^\nu_{[k]})$ and

$$\|x\| = \| \Delta^\nu_{[k]} x_k \|_\infty \text{ in } \ell_\infty(\Delta^\nu_{[k]}).$$

On the other hand we can show that the mapping $\triangle_T : T(\ell_\infty(\Delta^\nu_{[k]})) \to \ell_\infty$, defined by $\triangle_T(x) = (y_k) = (\Delta^\nu_{[k]} x_k)$ is a linear homomorphism.
Now

$$\|\triangle_T(x)\| = \|x\|.$$

Therefore, $\triangle_T$ is linear and bijective. Hence $(\ell_\infty(\Delta^\nu_{[k]})$ is isometrically isomorphic to ℓ_∞.

Corollary 8. *$\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$ and $c_0(\Delta_{[k]})$ are BK-spaces under the norm defined in (2).*

Theorem 9. *$c_0(\Delta^\nu_{[k]}) \subset c(\Delta^\nu_{[k]}) \subset \ell_\infty(\Delta^\nu_{[k]}) \subset \ell_\infty$ and the inclusion is strict.*

Proof. The proof is trivial.

Corollary 10. *$c_0(\Delta_{[k]}) \subset c(\Delta_{[k]}) \subset \ell_\infty(\Delta_{[k]}) \subset \ell_\infty$ and the inclusion is strict.*

Theorem 11. (i) *$\ell_\infty \cap \ell_\infty(\Delta_{[k]}) = \ell_\infty \cap c(\Delta_{[k]}) = c(\Delta_{[k]})$ and*

(ii) *$\ell_\infty \cap c_0(\Delta_{[k]}) = c_0(\Delta_{[k]})$.*

Proof. (i) The proof of $\ell_\infty \cap c(\Delta_{[k]}) = c(\Delta_{[k]})$ directly follows from Corollary 10 and only to show $\ell_\infty \cap \ell_\infty(\Delta_{[k]}) = c(\Delta_{[k]})$. Suppose $x \in \ell_\infty \cap \ell_\infty(\Delta_{[k]})$ which implies that $|x_k| < \infty$ and $|kx_k - (k+1)x_{k+1}| < \infty$ for all $k \in \mathbb{N}$. Thus, there exists ε_k and l such that $(kx_k - (k+1)x_{k+1}) = l + \varepsilon_k$, $\varepsilon_k \to 0$ as $k \to \infty$. Now,

$$\begin{aligned}\sum_{k=1}^{n}(kx_k - (k+1)x_{k+1}) &= x_1 - (n+1)x_{n+1} \\ &= nl + \sum_{k=1}^{n} \varepsilon_k.\end{aligned}$$

Therefore, $x \in c(\Delta_{[k]})$, this follows from the fact that $l = \frac{1}{n}x_1 - x_{n+1} - \frac{1}{n}x_{n+1} - \frac{1}{n}\sum_{k=1}^{n}\varepsilon_k$ and hence $(\ell_\infty \cap \ell_\infty(\Delta_{[k]})) \subset c(\Delta_{[k]})$. From Corollary 10, the fact $c(\Delta_{[k]}) \subset (\ell_\infty \cap \ell_\infty(\Delta_{[k]}))$ is clear. This completes the proof.

(ii) This follows from the Corollary 10.

3. Dual spaces

In this section, we give the definition of $p\alpha-, p\beta-$ and $p\gamma-$ duals of X, a nonempty subset of ω and determine these duals for the spaces $\ell_\infty(\Delta^\nu_{[k]}), c(\Delta^\nu_{[k]})$ and $c_0(\Delta^\nu_{[k]})$. We also discuss some lemmas and theorems associated to this concept.

Definition 1. *Let X be a nonempty subset of ω and $p \geq 1$, then*

$$\begin{aligned} X^{p\alpha} &= \Big\{(y_k) \in \omega : \sum_k |x_k y_k|^p < \infty \textit{ for every } x \in X \Big\}, \\ X^{p\beta} &= \Big\{(y_k) \in \omega : \sum_k (x_k y_k)^p \textit{ converges for every } x \in X \Big\}, \\ X^{p\gamma} &= \Big\{(y_k) \in \omega : \sup_{M\in\mathbb{N}} \Big|\sum_{k=1}^{M} (x_k y_k)^p\Big| < \infty \textit{ for every } x \in X \Big\}. \end{aligned}$$

We call $X^{p\alpha}, X^{p\beta}$ and $X^{p\gamma}$ are the $p\alpha-, p\beta-$ and $p\gamma-$ duals of X, respectively. For $p = 1$, X^α is called the *Köthe−Toeplitz* dual of X. It is clear that $X^{p\alpha} \subset X^{p\beta} \subset X^{p\gamma}$ and for $X \subset Y$, $X^\eta \subset X^\eta$, where $\eta \in \{p\alpha, p\beta, p\gamma\}$. The concept of duality of the sequence spaces was introduced by Köthe[14]. Furthermore, this concepts were extended and studied by Maddox[15, 16], Kamthan and Gupta[17] , Malkowsky et al. [11], Et and Esi [3], and many others.

Lemma 12. $\sup_k |\Delta^\nu_{[k]} x_k| < \infty$ *if and only if*

$$\begin{aligned} &(i) \qquad \sup_k |\nu_k x_k| < \infty. \\ &(ii) \qquad \sup_k |\nu_k x_k - \frac{k+1}{k}\nu_{k+1}x_{k+1}| < \infty. \end{aligned}$$

Proof. For necessity, let $\sup_k |\Delta^\nu_{[k]} x_k| < \infty$,

$$\text{i.e.} \qquad |k\nu_k x_k - (k+1)\nu_{k+1}x_{k+1}| < \infty, \quad \text{for all } k = 1, 2, 3...$$

Consider,

$$\begin{aligned} |\nu_1 x_1 - k\nu_k x_k| &= |\sum_{i=1}^{k-1}(i\nu_i x_i - (i+1)\nu_{i+1}x_{i+1})| \\ &\leq \sum_{i=1}^{k-1}|i\nu_i x_i - (i+1)\nu_{i+1}x_{i+1}| \\ &< O(k) \\ \Rightarrow \quad |\nu_k x_k| &< O(1), \quad \text{for all } k = 1, 2, 3.... \end{aligned}$$

For the second part,

$$\begin{aligned} \Big|\nu_k x_k &- \frac{k+1}{k}\nu_{k+1}x_{k+1}\Big| \\ &= \frac{1}{k}|k\nu_k x_k - (k+1)\nu_{k+1}x_{k+1}| \\ &\leq \frac{1}{k}\Big[|k\nu_k x_k - (k+1)\nu_{k+1}x_{k+1}| < \infty. \end{aligned}$$

For sufficiency, suppose (i) and (ii) hold, then $\sup_{k\geq 1}|\Delta_{[k]}(x_k\nu_k)|^{p_k} < \infty$ due to the fact that

$$\begin{aligned} \Big|\nu_k x_k &- \frac{(k+1)}{k}\nu_{k+1}x_{k+1}\Big| \\ &\geq \frac{1}{k}\Big[|k\nu_k x_k - (k+1)\nu_{k+1}x_{k+1}| - |\nu_k x_k|\Big]. \end{aligned}$$

This completes the proof.

Lemma 13. $\sup_k |\Delta_{[k]}x_k| < \infty$ *if and only if*

$$(i) \qquad \sup_k |x_k| < \infty.$$

$$(ii) \qquad \sup_k |x_k - \frac{k+1}{k}x_{k+1}| < \infty.$$

Proof. Proof is similar to that of Lemma 12.

Theorem 14.

$$\Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{p\alpha} = \Big[c(\Delta^\nu_{[k]})\Big]^{p\alpha} = \Big[c_0(\Delta^\nu_{[k]})\Big]^{p\alpha} = D_1,$$

$$\textit{where} \quad D_1 = \bigcap_{1<N\in\mathbb{N}}\Big\{x = (x_k) : \sum_k N^p|\nu_k^{-1}x_k|^p < \infty\Big\}.$$

Proof. For first inclusion, let $x \in D_1$ and $y \in \ell_\infty(\Delta^\nu_{[k]})$. By Lemma 12, there exists a positive integer N such that

$$|y_k \nu_k| < N, \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, $\sum_k |x_k y_k|^p \leq \sum_k |x_k|^p N^p |\nu_k^{-1}|^p \leq N^p \sum_k |\nu_k^{-1} x_k|^p < \infty.$

Since $x \in D_1$, the series on the right hand side of the above inequality is less than ∞, which implies $x \in \Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{p\alpha}$.

For the second part, let $x \in \Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{p\alpha}$ and $x \notin D_1$.

Then there exists a positive integer $N > 1$ such that

$$\sum_k N^p |\nu_k^{-1} x_k|^p = \infty.$$

Now we define a sequence $y = (y_k)$ such that

$$y_k = \frac{N}{\nu_k} \cdot \text{sgn } x_k; \quad k = 1, 2, 3 \ldots$$

Then it is easy to verify $y \in \ell_\infty(\Delta^\nu_{[k]})$, but $\sum_k |x_k y_k|^p = \infty$.

This contradicts the assumption that $x \in \Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{p\alpha}$. Proofs of other spaces are similar.

Corollary 15.

$$\Big[\ell_\infty(\Delta_{[k]})\Big]^\alpha = \Big[c(\Delta_{[k]})\Big]^\alpha = \Big[c_0(\Delta_{[k]})\Big]^\alpha = D_1',$$
$$\textit{where} \quad D_1' = \bigcap_{1<N\in\mathbb{N}} \Big\{x = (x_k) : \sum_k N|x_k| < \infty\Big\}.$$

Proof. The proof of this corollary can be obtained by putting $\nu_k = 1$ for all $k \in \mathbb{N}$ and $p = 1$ in the Theorem 14.

Theorem 16. *uppose η stands for $p\alpha-, p\beta-$and $p\gamma-$ duals, then*

$$\Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{\eta\eta} = \Big[c(\Delta^\nu_{[k]})\Big]^{\eta\eta} = \Big[c_0(\Delta^\nu_{[k]})\Big]^{\eta\eta} = D_2,$$
$$\textit{where} \quad D_2 = \bigcup_{1<N\in\mathbb{N}} \Big\{x = (x_k) : \sup_k \frac{|\nu_k x_k|^p}{N^p} < \infty\Big\}.$$

Proof. For the first part, suppose $x \in D_2$ and $\eta = p\alpha$, then

$$\frac{|\nu_k x_k|^p}{N^p} < \infty, \quad \text{for all} \quad k = 1, 2, 3..$$

Let $y \in D_1$ and by Theorem 14,

$$\sum_k |x_k y_k|^p \leq \sup_k \frac{|\nu_k x_k|^p}{N^p} \sum_k N^p |\nu_k^{-1} y_k|^p < \infty, \quad \text{for all} \quad k \in \mathbb{N}.$$

which implies $x \in \Big[D_1\Big]^{p\alpha} = \Big[\Big[\ell_\infty(\Delta^\nu_{[k]})\Big]^{p\alpha}\Big]^{p\alpha}$.
For the second part, let $x \in D_1$ and $x \notin D_2$.
Then, there exists a positive integer $N > 1$ such that

$$\sup_k \frac{|\nu_k x_k|^p}{N^p} = \infty.$$

Hence there exists a positive increasing sequence $(k(i))$ such that

$$\frac{|\nu_{k(i)} x_{k(i)}|^p}{N^p} > i^{p+k} \quad \text{for all } k \in \mathbb{N}.$$

Now we define a sequence $y = (y_k)$ such that

$$y_k = \begin{cases} |x_{k(i)}|^{-p} & k = k(i), \\ 0 & \text{otherwise,} \end{cases}$$

Now,

$$\sum_k \frac{N^p |y_k|^p}{|\nu_k|^p} \leq \sum_{i=1}^{\infty} \frac{N^p |x_{k(i)}|^{-p}}{|\nu_{k(i)}|^p} < \sum_i^{\infty} i^{-(k+p)} < \infty, \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, $y \in D_1$, but $\sum_k |x_k y_k|^p = \infty$.
This contradicts the assumption that $x \in D_1$. Proofs for other spaces are obtained by using similar techniques.

Corollary 17.

$$\Big[\ell_\infty(\Delta_{[k]})\Big]^{\eta\eta} = \Big[c(\Delta_{[k]})\Big]^{\eta\eta} = \Big[c_0(\Delta_{[k]})\Big]^{\eta\eta} = D_2',$$

$$\textit{where} \quad D_2' = \bigcup_{1<N\in\mathbb{N}} \Big\{x = (x_k) : \sup_k \frac{|x_k|}{N} < \infty\Big\}.$$

4. [k]-statistical convergence

In this section, we give the definition of [k] -statistical convergence and establish some relations between the spaces defined by us and othe r spaces. The notion of statistical convergence was introduced by Fast [18] and studied by various authors such as Fridy [19] Connor[20], Kolk[21], Et. and Nuray [4] and Mursaleen [22].
We recall some concepts connecting with statistical convergence. Let K be a subset of $\mathbb{N}$, the set of natural numbers and K_n be a set i.e.

$$K_n = \{k \in K : k < n\},$$

then the natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, provided the limit exists, where $|K_n|$ denotes the number of elements in K_n. Finite subsets have natural density zero.

Definition 2. *A sequence $x = (x_k)$ is said to be statistically convergent or S-convergent to L, if for every $\epsilon > 0$*

$$\lim_{m\to\infty} \frac{1}{m}|\{k < m : |x_k - L| \geq \epsilon\}| = 0.$$

In other words the natural density of the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ i.e. $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0$. In this case we write $St. - \lim x = L$ or $x_k \to L(S)$ and

$$S = \{x \in \omega : St. - \lim x = L, \text{for some } L\}.$$

Definition 3. *A sequence $x = (x_k)$ is said to be [k]-statistically convergent or $S(\Delta^{\nu}_{[k]})$-convergent to L, if for every $\epsilon > 0$*

$$\lim_{m\to\infty} \frac{1}{m}|\{k < m : |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}| = 0.$$

In this case we write $\delta(\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} x - L| \geq \epsilon\}) = 0, \quad St. - \lim x = L$ or $x_k \to LS(\Delta^{\nu}_{[k]})$

Theorem 18. *Let $x = (x_k)$ be a sequence and [k]-statistically convergent to L in $S(\Delta^{\nu}_{[k]})$, then L is unique.*

Proof. The proof is trivial, hence omitted.

Theorem 19. *Let (x_k) be a sequence and (y_k) be a [k]-statistically convergent sequence such that $x_k = y_k$ almost all k, then (x_k) is a [k]-statistically convergent sequence.*

Proof. Suppose $x_k = y_k$ almost all k, then $\delta(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$,
Let $y_k \to LS(\Delta^{\nu}_{[k]})$, then for every $\epsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} y_k - L| \geq \epsilon\}) = 0,$$

Now, $\delta(\{k \in I_m : |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\})$

$$\begin{aligned} &\leq \delta(\{k \in \mathbb{N} : x_k = y_k \text{ and } |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}) \\ &\quad + \delta(\{k \in \mathbb{N} : x_k \neq y_k \text{ and } |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}) \\ &= \delta(\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} y_k - L| \geq \epsilon\}) + 0 = 0, \end{aligned}$$

$\Rightarrow \quad x_k \to LS(\Delta^{\nu}_{[k]})$.

Theorem 20. **(i)** *If* $x_k \to Lw(\Delta^{\nu}_{[k]})$, *then* $x_k \to LS(\Delta^{\nu}_{[k]})$,

(ii) *If* $x \in c(\Delta^{\nu}_{[k]})$ *and* $x_k \to LS(\Delta^{\nu}_{[k]})$, *then* $x_k \to Lw(\Delta^{\nu}_{[k]})$,

(iii) $S(\Delta^{\nu}_{[k]}) \cap c(\Delta^{\nu}_{[k]}) = w(\Delta^{\nu}_{[k]}) \cap c(\Delta^{\nu}_{[k]})$,

where $w(\Delta^{\nu}_{[k]}) = \Big\{x = (x_k) \in \omega : \dfrac{1}{m}\displaystyle\sum_{k=1}^{m} |\Delta^{\nu}_{[k]} x_k - L| = 0, \ \textit{for some } L\Big\}$.

Proof. **(i)** Let $x_k \to Lw(\Delta^{\nu}_{[k]})$, this implies for every $\epsilon > 0$,

$$\begin{aligned} \frac{1}{m}\sum_{k=1}^{m} |\Delta^{\nu}_{[k]} x_k - L| &\geq \sum_{\substack{k\in\mathbb{N}, \\ |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon}} |\Delta^{\nu}_{[k]} x_k - L| \\ &\geq \epsilon |\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}| \end{aligned}$$

Taking limit as $n \to \infty$, $|\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}| = 0$ which implies $x_k \to LS(\Delta^{\nu}_{[k]})$.

(ii) Suppose $x \in (\Delta^{\nu}_{[k]})$ and $x_k \to LS(\Delta^{\nu}_{[k]})$. i.e, for given $\epsilon > 0$, $|\Delta^{\nu}_{[k]} x_k - L| < \epsilon$ as $k \to \infty$. Now,

$$\begin{aligned} \frac{1}{m}\sum_{k=1}^{m} |\Delta^{\nu}_{[k]} x_k - L| &= \frac{1}{m} \sum_{\substack{k\in\mathbb{N}, \\ |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon}} |\Delta^{\nu}_{[k]} x_k - L| + \\ &\quad \frac{1}{m} \sum_{\substack{k\in\mathbb{N}, \\ |\Delta^{\nu}_{[k]} x_k - L| < \epsilon}} |\Delta^{\nu}_{[k]} x_k - L| \\ &\leq \frac{1}{m} |\{k \in \mathbb{N} : |\Delta^{\nu}_{[k]} x_k - L| \geq \epsilon\}| + \epsilon. \end{aligned}$$

As $m \to \infty$, the right hand side is zero, which implies that $x_k \to Lc(\Delta^{\nu}_{[k]})$.

(iii) This immediately follows from (i) and (ii).

Theorem 21. If $\liminf_m \frac{\lambda_m}{m} > 0$, then $S(\hat{A}, \Delta^r_\nu) \subset S_\lambda(\hat{A}, \Delta^r_\nu)$.

Proof. Given $\epsilon > 0$, we have

$$\{k \in I_m : |\Delta^r_\nu B_{kn}(x) - L| \geq \epsilon\} \subset \{k \leq m : |\Delta^r_\nu B_{kn}(x) - L| \geq \epsilon\}$$

Therefore, $\frac{1}{m}|\{k \leq m : |\Delta^r_\nu B_{kn}(x) - L| \geq \epsilon\}| \geq \frac{1}{m}|\{k \in I_m : |\Delta^r_\nu B_{kn}(x) - L| \geq \epsilon\}|$

$$= \frac{\lambda_m}{m}.\frac{1}{\lambda_m}|\{k \in I_m : |\Delta^r_\nu B_{kn}(x) - L| \geq \epsilon\}|$$

Taking the limit as $m \to \infty$ we get $x_k \to LS(\hat{A}, \Delta^r_\nu) \quad \Rightarrow \quad x_k \to LS_\lambda(\hat{A}, \Delta^r_\nu)$.

5. Matrix transformations

Let X and Y be any two subspaces of ω. By (X, Y), we denote all the matrix transformations from X to Y. Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} such that $A : X \to Y$, defined by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k, \quad (n \in \mathbb{N}) \tag{5}$$

where $x \in X$ and $A(x)$ denotes the sequence $(A(x))_n$ provided the sum in (5) is convergent. Before proceed to the main theorems, first we give some known results concerning matrix transformations.

We take a list of results such as

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty, \tag{6}$$

$$\lim_{n\to\infty} a_{nk} = a_k, \quad \text{for} \quad k \in \mathbb{N}, \tag{7}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{\infty} |a_k|, \tag{8}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = a, \tag{9}$$

$$\sup_n \left| \sum_{k=1}^{\infty} a_{nk} \right| < \infty, \tag{10}$$

$$\sup_n \sum_{k=1}^{\infty} |k a_{nk}| < \infty, \tag{11}$$

$$\sup_{n,k} |a_{nk}| < \infty, \tag{12}$$

$$\lim_{n\to\infty} a_{nk} = 0, \quad \text{for} \quad k \in \mathbb{N}, \tag{13}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0, \tag{14}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |k a_{nk}| = 0, \tag{15}$$

Lemma 22. ([23])

(a) *For $X = \{\ell_\infty, c, c_0\}$, $A \in (X, \ell_\infty)$ if and only if condition (6) holds.*

(b) *$A \in (\ell_\infty, c)$ if and only if conditions (7) and (8) hold.*

(c) *$A \in (c, c)$ if and only if conditions (6), (7) and (9) hold.*

(d) *$A \in (c_0, c)$ if and only if conditions (6) and (7) hold.*

(e) *For $X = \{\ell_\infty, c, c_0\}$, $A \in (X, c_0)$ if and only if conditions (13) and (14) hold.*

Now we obtain necessary and sufficient conditions for the matrix transformations of $\ell_\infty(\Delta_{[k]}), c(\Delta_{[k]})$, $c_0(\Delta_{[k]})$ to ℓ_∞, c ,c_0 and vice versa.

Theorem 23. (i) *$A \in (\ell_\infty(\Delta_{[k]}), \ell_\infty)$ if and only if condition (10) holds.*

(ii) $A \in (c(\Delta_{[k]}), \ell_\infty)$ *if and only if conditions (6) and (10) hold.*

(iii) $A \in (c_0(\Delta_{[k]}), \ell_\infty)$ *if and only if conditions (6) and (10) hold.*

(iv) $A \in (\ell_\infty(\Delta_{[k]}), c)$ *if and only if condition (6) and (8) hold.*

(v) $A \in (c(\Delta_{[k]}), c)$ *if and only if conditions (6), (8) and (10) hold.*

(vi) $A \in (c_0(\Delta_{[k]}), c)$ *if and only if conditions (6), (8) and (10) hold.*

(vii) $A \in (\ell_\infty(\Delta_{[k]}), c_0)$ *if and only if condition (10) holds.*

(viii) $A \in (c(\Delta_{[k]}), c_0)$ *if and only if conditions (10) and (12) hold.*

(ix) $A \in (c_0(\Delta_{[k]}), c_0)$ *if and only if conditions (10) and (12) hold.*

Proof. **(i)** Sufficiency: Suppose $x \in \ell_\infty(\Delta_{[k]})$, by Lemma 12, there exists a real M such that $\sup_k |x_k| < M$, for all k. Now

$$\sup_n |A_n(x)| = \sup_n \left| \sum_k a_{nk} x_k \right| \leq \sup_n \left| \sum_k a_{nk} \right| \sup_k |x_k|$$
$$\leq M \sup_n \left| \sum_k a_{nk} \right| < \infty, \quad \text{by the condition (10).}$$

Necessity: Suppose $\sup_n |A_n(x)| < \infty$, by putting $x = e = (1, 1, 1, ...)$, we have

$$\sup_n |A_n(e)| = \sup_n \left| \sum_k a_{nk} \right| < \infty.$$

(ii) Sufficiency: Suppose $x \in c(\Delta_{[k]})$, there exists a $l \in \mathbb{C}$ such that

$$kx_k - (k+1)x_{k+1} = l + \varepsilon_k, \quad \varepsilon_k \to 0 \ \text{ as } k \to \infty.$$

We have

$$x_1 - kx_k = \sum_{i=1}^{k-1}(l+\varepsilon_i) = l\sum_{i=1}^{k-1} + \sum_{i=1}^{k-1}\varepsilon_i$$

$$\Rightarrow \qquad x_k = \frac{x_1}{k} - l\frac{(k-1)}{2} + \frac{1}{k}\sum_{i=1}^{k-1}\varepsilon_i$$

Now

$$\sup_n |A_n(x)| = \sup_n \left|\sum_k a_{nk}\left(\frac{x_1}{k} - l\frac{(k-1)}{2} + \frac{1}{k}\sum_{i=1}^{k-1}\varepsilon_i\right)\right|$$

$$\leq |x_1|\sup_n\left|\sum_k \frac{a_{nk}}{k}\right| + \frac{|l|}{2}\sup_n\left|\sum_k a_{nk}(k-1)\right| + \frac{\varepsilon_M}{2}\sup_n\left|\sum_k a_{nk}(k-1)\right|$$

$$\leq \left(|x_1| + \frac{|l|}{2} + \frac{\varepsilon_M}{2}\right)\sup_n\left|\sum_k a_{nk}\right| + \left(\frac{|l|}{2} + \frac{\varepsilon_M}{2}\right)\sup_n\left|\sum_k k a_{nk}\right|$$

$$< \infty, \quad \text{by the conditions (10) and (11),}$$

where $\varepsilon_M = max(0, \sup_k |\varepsilon_k|)$.

Necessity: Necessity of the conditions (10) and (11) can be obtained by taking $x = e$ and $x_k = k$ for all k, respectively in the hypothesis

$$\sup_n |A_n(x)| = \sup_n \left|\sum_k a_{nk}x_k\right| < \infty.$$

(iii) The proof is immediate by putting $l = 0$ in (ii).

(iv) Sufficiency: Suppose $x \in \ell_\infty(\Delta_{[k]})$ and the condition (12) holds, by Lemma 12, we have

$$\lim_{n\to\infty} |A_n(x)| = \lim_{n\to\infty}\left|\sum_k a_{nk}x_k\right| \leq M \lim_{n\to\infty}\left|\sum_k a_{nk}\right| < \infty.$$

Necessity: Suppose $\lim_{n\to\infty}|A_n(x)| < \infty$, by putting $x = e = (1,1,1,.....)$, we have

$$\lim_{n\to\infty} |A_n(e)| = \lim_{n\to\infty}\left|\sum_k a_{nk}\right| < \infty.$$

(v) The proof follows from (i), (ii) and (iv).

(vi) The proof follows from (v) and (iii).

(vii) The proof follows from Lemma 22 and (i).

(viii) The proof follows from Lemma 22, (i) and (ii).

(ix) The proof follows from (viii).
This completes the proof.

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