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PRIME BCK- SUBMODULES OF BCK- MODULES

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Abstract. In this paper by considering the notion of BCK-module X, we have introduced prime BCK- submodules and we have proved some results by it. As a result we have shown that if M_1 and M_2 be left BCK- modules over X and ϕ be a BCK- epimorphism from M_1 to M_2. Also N be a prime BCK- submodule of M_2. Then $\phi^{-1}(N)$ is a prime BCK- submodule of M_1.

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1. Introduction

Every module is an action of ring on certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. BCK-module is an action of BCK-algebra on commutative group. In 1994, the notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem [2]. They established isomorphism theorems and studied some properties of BCK-modules. The theory of BCK-modules was further developed by Z. Perveen and M. Aslam [9]. Now, in this paper we have introduced the concept of prime BCK-submodules and we have proved some results by it. As a result we have shown that if M_1 and M_2 be left BCK- modules over X and ϕ be a BCK- epimorphism from M_1 to M_2. Also N be a prime BCK- submodule of M_2. Then $\phi^{-1}(N)$ is a prime BCK- submodule of M_1.

2. Preliminaries

Let us to begin this section with the definition of a BCK-algebra.

Definition 1. *[8] Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BCK- algebra if it satisfies the following axioms:*
(BCK1)$((x * y) * (x * z)) * (z * y) = 0$,

(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = y * x = 0$ *imply that* $x = y$, *for all* $x, y, z \in X$.
We can define a partial ordering $\leq$ *by* $x \leq y$ *if and only if* $x * y = 0$.
If there is an element 1 of a BCK- *algebra* X, *satisfying* $x * 1 = 0$, *for all* $x \in X$, *the element 1 is called unit of* X. *A* BCK- *algebra with unit is called to be bounded.*

Definition 2. *[8] Let* $(X, *, 0)$ *be a* BCK- *algebra and* X_0 *be a nonempty subset of* X. *Then* X_0 *is called to be a subalgebra of X, if for any* $x, y \in X_0$, $x * y \in X_0$ *i.e.,* X_0 *is closed under the binary operation* $*$ *of X.*

Definition 3. *[8] A* BCK- *algebra* $(X, *, 0)$ *is said to be commutative , if it satisfies,* $x * (x * y) = y * (y * x)$, *for all* x, y *in X.*

Definition 4. *[8] A* BCK- *algebra* $(X, *, 0)$ *is called implicative, if* $x = x * (y * x)$, *for all* x, y *in X.*

Theorem 1. *[8] Every implicative* BCK*-algebra is a commutative, but its converse may not be true.*

Definition 5. *[8] A non-empty subset A of* BCK- *algebra* $(X, *, 0)$ *is called an ideal of X if it satisfies the following conditions:*
(i) $0 \in A$,
(ii) $(\forall x \in X)(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

Theorem 2. *[2] Let X be a bounded implicative* BCK- *algebra and let* $x + y = (x * y) \vee (y * x)$, *for all* $x, y \in X$ *then we have:*
(i) $(X, +)$ *forms a commutative group,*
(ii) Any ideal I of X consisting of two elements forms an X- module.

Definition 6. *[8] Suppose A is an ideal of* BCK- *algebra* $(X, *, 0)$. *For any* x, y *in X, we denote* $x \sim y$ *if and only if* $x * y \in A$ *and* $y * x \in A$. *It is easy to see that,* $\sim$ *is an equivalence relation on X.*
Denote the equivalence class containing x *by* C_x *and* $\frac{X}{A} = \{C_x : x \in X\}$. *Also we define* $C_x * C_y = C_{x*y}$, *for all* x, y *in* X .

Definition 7. *[8] Let X be a lower* BCK- *semilattice and A be a proper ideal of X. Then A is said to be prime if* $a \wedge b = b * (b * a) \in A$ *implies that* $a \in A$ *or* $b \in A$, *for any* a, b *in X.*

Theorem 3. *[8] In a lower* BCK- *semilattice* $(X, *, 0)$ *the following are equivalent:*
(i) I is a prime ideal,
(ii) I is an ideal and satisfies that for any $A, B \in I(X)$, $A \subseteq I$ *or* $B \subseteq I$ *whenever* $A \cap B \subseteq I$.

Definition 8. *[1] Let* $(X, *, 0)$ *be a* BCK*-algebra,* M *be an abelian group under* $+$ *and let* $(x, m) \longrightarrow x \cdot m$ *be a mapping of* $X \times M \longrightarrow M$ *such that*
(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = 0$,
for all $x, y \in X, m_1, m_2 \in M$,*where* $x \wedge y = y * (y * x)$. *Then* M *is called a left* X*-module.*
If X *is bounded, then the following additional condition holds:*
(iv) $1 \cdot m = m$.

A right X-module can be defined similarly.

Theorem 4. *[1] Every bounded implicative* BCK*-algebra forms module over itself.*
In the sequel X *is a* BCK*-algebra.*

Example 1. *[1] Let* A *be a non-empty set and* $X = P(A)$ *be the power set of* A. *Then* X *is a bounded commutative* BCK*-algebra with* $x \wedge y = x \cap y$, *for all* $x, y \in X$. *Define* $x + y = (x \cup y) \cap (x \cap y)'$, *the symmetric difference. Then* $M = (X, +)$ *is an abelian group with empty set* $\emptyset$ *as an identity element and* $x + x = \emptyset$. *Define* $x \cdot m = x \cap m$, *for any* $x, m \in X$. *Then simple calculations show that :*
(i) $(x \wedge y) \cdot m = (x \cap y) \cap m = x \cap (y \cap m) = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = \emptyset \cap m = \emptyset = 0$,
(iv) $1 \cdot m = A \cap m = m$. *Thus* X *itself is an* X*-module.*

Definition 9. *[1] Let* M_1, M_2 *be* X*-modules. A mapping* $f : M_1 \longrightarrow M_2$ *is called a* BCK*- homomorphism, if for any* $m_1, m_2 \in M_1$, *we have :*
(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$,
(ii) $f(x \cdot m_1) = x \cdot f(m_1)$, *for all* $x \in X$.
Ker(f) *and Img*(f) *have usual meaning.*

Definition 10. *[4] Let* $(X, *, 0)$ *be a* BCK*-algebra,* M *be an abelian group under* $+$ *and let* $(x, m) \longrightarrow x \cdot m$ *be a mapping of* $X \times M \longrightarrow M$ *such that*
(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = 0$,
(iv) $(x \vee y).m = x.m + (y * x).m$.
then M *is called an extended* BCK*-module.*

Definition 11. *Let* M *be a left* BCK*- module over* X*, and* N *be a* BCK*- submodule of* M*, then we define* $Ann_X(M) = \{x \in X \mid x \cdot m = 0, \text{ for all } m \in M\}$. M *is called faithful if* $Ann_X(M) = 0$.

Theorem 5. *[2] Any ideal consisting of two elements in a bounded commutative BCK- algebra X forms an X- module under the binary operations $x.m = x \wedge m$.*

Example 2. *[4] Let X be a non-empty set. Then $(P(X), -)$ is a bounded BCK-algebras, Z(integer set) with the followings operations is a $P(X)$-module, $x_0 \in X$ and $\cdot : P(X) \times Z \to Z$ such that*

$$A.n = \begin{cases} n & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$$

3. Prime BCK- submodule

The notion of BCK-module was introduced by Abujabal, Aslam and Thaheem [1]. A BCK-module is an action of a BCK-algebra on abelian group $(M, +)$. In this section we have defined prime BCK-submoduls and have obtained some theorems.

Definition 12. *Let M be a left BCK- module over X and N be a submodule of M. Then N is said to be prime BCK-submodule of M, if $N \neq M$ and $x \cdot m \in N$, implies that $m \in N$ or $x.M \subseteq N$, for any x in X and any m in M.*

Example 3. *Let $X = P(A = \{1, 2, ..., n\})$, $B_i = \{1, 2, ..., n\} - \{i\}$, for $i \in \{1, 2, ..., n\}$. Then $P(B_i)$ is a prime BCK- submodule of $P(A)$, because we can define*
$\cdot = \cap : P(A) \times P(B_i) \longrightarrow P(B_i)$. It is easy to see that $P(B_i)$ is a BCK- submodule of $P(A)$. Now we show that $P(B_i)$ is a prime BCK-submodule. Let for subsets C and D of A, $C \cap D \subseteq P(B_i)$, $D \notin P(B_i)$ and $C \cap P(A) \nsubseteq P(B_i)$. Then $i \in D$ and there exists $K \subseteq A$ such that $C \cap K \nsubseteq B_i$. Since $B_i = \{1, 2, ..., n\} - \{i\}$, therefore $i \in C \cap K$. So $i \in D \cap C \cap K \subseteq D \cap C \subseteq B_i$ and this is a contradiction. Then $P(B_i)$ is a prime BCK- submodule of $P(A)$.

Theorem 6. *Let M be a left BCK-module over X. Then P is a prime BCK-submodule of M containing N if and only if $\frac{P}{N}$ is a prime BCK-submodule of $\frac{M}{N}$.*

Proof. Necessity. First we show that $\frac{P}{N} \neq \frac{M}{N}$. Since P is a prime BCK-submodule of M , then $N \neq M$ therefore there exists $m \in M - P$, so $m + N \in \frac{M}{N} - \frac{P}{N}$. In fact if $m + N \in \frac{P}{N}$, then $m + N = p_1 + N$ for some $p_1 \in P$, hence $m - p_1 \in N \subseteq P$ and so $m \in P$, which is a contradiction.
Let $(x, m + N) \longrightarrow x \cdot m + N$ be a mapping of $X \times \frac{M}{N} \longrightarrow \frac{M}{N}$.
Now let $x \in X$ and $m \in M$ such that $x \cdot (m + N) \in \frac{P}{N}$ i.e. $x \cdot m + N \in \frac{P}{N}$, then $x \cdot m + N = p_1 + N$, for some $p_1 \in P$, $x \cdot m - p_1 \in N \subseteq P$. So $x \cdot m \in P$. Since P

is a prime BCK- submodule P we get that $m \in P$ or $x \cdot M \subseteq P$. If $m \in P$, then $m+N \in \frac{P}{N}$ and the proof is complete. If $x \cdot M \subseteq P$, then for all $m \in M, x \cdot m+N \in \frac{P}{N}$ i.e. $x \cdot (m+N) \in \frac{P}{N}$. Hence $x \cdot \frac{M}{N} \subseteq \frac{P}{N}$.
Sufficiency. First we show that $P \neq M$. we have $\frac{P}{N} \neq \frac{M}{N}$, so there exists $m \in M$, such that $m+N \notin \frac{P}{N}$. We claim $m \notin P$. If $m \in P$, hence $m+N \in \frac{P}{N}$, and this is a contradiction. Now let $x \in X$ and $m \in M$ such that $x \cdot m \in P$, clearly $x \cdot m+N \in \frac{P}{N}$, for all $m \in M$. Since $\frac{P}{N}$ is a prime BCK- submodule of $\frac{M}{N}$. So $m+N \in \frac{P}{N}$ or $x \cdot \frac{M}{N} \subseteq \frac{P}{N}$.
If $m+N \in \frac{P}{N}$, then $m+N = p_1+N$, for some $p_1 \in P$, hence $m-p_1 \in N \subseteq P$, then $m \in P$ and the proof is complete. If $x \cdot \frac{M}{N} \subseteq \frac{P}{N}$, then $x \cdot (m+N) \in \frac{P}{N}$ for all $m \in M$, so $x \cdot m+N \in \frac{P}{N}$. Since $N \subseteq P$, we get that $x \cdot m \in P$, for all $m \in M$ i.e. $x \cdot M \subseteq P$. Therefore the proof is complete.

Theorem 7. *In Example 1, let I be a prime ideal of X . Then $P(I)$ is a prime BCK- submodule of $P(X)$.*

Proof. Since $I \neq X$, then $P(I) \neq P(X)$. Now let K and N be subsets of X and $K \wedge N = K \cap N \in P(I)$. Since I is a prime ideal of X, then $K \subseteq I$ or $N \subseteq I$. If $N \subseteq I$, the proof is complete. If $K \subseteq I$, we have for all $C \subseteq X, K \cap C \subseteq K \subseteq I$ i.e. $K \cap C \subseteq I$ and this complete the proof.

In the sequel X is a BCK-algebra.

Definition 13. *A left BCK-module M over X, will be called fully faithful, if every nonzero BCK- submodule of M is faithful.*

Remark 1. *Let M be a left BCK- module over X and N be a BCK- submodule of M. Then we define $(N : M) = \{x \in X \mid x \cdot M \subseteq N\}$.*

Theorem 8. *Let X be a bounded implicative BCK- algebra and M be an extended X-module. BCK- submodule N of M, is prime if and only if, $P = (N : M)$ is a prime ideal of X and the left $\frac{X}{P}$- module $\frac{M}{N}$ is fully faithful.*

Proof. Necessity. Suppose N is a prime BCK- submodule of M. Now we prove that $(N : M)$ is a prime ideal of X. By primitivity N, we have $(N : M) \neq X$, because $1 \in X$, but $1 \notin (N : M)$. Now we show that $(N : M)$ is a prime ideal. Let $(x \wedge y) \in (N : M)$, for all $x, y \in X$, so $(x \wedge y) \cdot M \subseteq N$, therefore $(x \wedge y) \cdot m = x \cdot (y \cdot m) \in N$, for all $m \in M$. Since $x \in X$ and N is a prime BCK- submodule of M, then $y \cdot m \in N$ or $x \cdot M \subseteq N$.
If $x \cdot M \subseteq N$, then $x \in (N : M)$.
If $x \cdot M \nsubseteq N$, we show that $y \cdot M \subseteq N$.
Because if $y \cdot M \nsubseteq N$, then there exists $m_1 \in M$ such that $y \cdot m_1 \notin N$. Since

$x \cdot (y \cdot m) \in N$, for all $m \in M$, then $x \cdot (y \cdot m_1) \in N$. By primitivity N, we get $x \cdot M \subseteq N$, this is a contradiction. Hence $y \cdot M \subseteq N$. So $P = (N : M)$ is a prime ideal of X. Since N is prime, then $N \neq M$. So there exists $m_0 \in M - N$. Now we show that the left $\frac{X}{P}$-module $\frac{M}{N}$ is fully faithful. Since $x.m + N = N$ for all $m \in M$, then $x.m \in N$. So $x.m_0 \in N$. By primitivity N, $m_0 \in N$ or $x.M \subseteq N$. Since $m_0 \notin N$, then $x.M \subseteq N$. Hence $x \in (N : M) = P$. Then every submodule of $\frac{M}{N}$ is faithful. So $\frac{X}{P}$-module $\frac{M}{N}$ is fully faithful.
Sufficiency. let for any $x \in X$ and $m \in M$, $x \cdot m \in N$. Then it is easy to see that $\frac{<m>+N}{N}$, is $\frac{X}{P}$- BCK- submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ is a fully faithful $\frac{X}{P}$ module and $(x+P)\cdot(< m > +N) = x\cdot < m > +N = N$, then $x+P = P$ i.e. $x \in P = (N : M)$. Hence $x \cdot M \subseteq N$. Therefore N is a prime BCK- submodule of M.

Theorem 9. *Let M_1 and M_2 be left BCK- modules over X and ϕ be a BCK-epimorphism from M_1 to M_2. Also N be a prime BCK- submodule of M_2. Then $\phi^{-1}(N)$ is a prime BCK- submodule of M_1.*

Proof. It is immediate that $\phi^{-1}(N) \neq M_1$, now we show that $\phi^{-1}(N)$ is a prime BCK- submodule of M_1. Let $x \in X$ and $m \in M_1$ such that $x \cdot m \in \phi^{-1}(N)$, so $\phi(x \cdot m) \in N$, hence $x \cdot \phi(m) \in N$, since N is a prime BCK- submodule of M. Therefore $x \cdot M_2 \subseteq N$ or $\phi(m) \in N$. If $x \cdot M_2 \subseteq N$, then it is easy to see that $x \cdot M_1 \subseteq \phi^{-1}(N)$, also if $\phi(m) \in N$, so $m \in \phi^{-1}(N)$. This complete the proof.

In above theorem, it may be N a prime BCK- submodule of M_1, but $\phi(N)$ is not a prime BCK- submodule of M_2.
Consider the following example:

Example 4. *In Example 3, let $A = \{1, 2\}$ and $B = \{1\}$, and let $\lambda : P(A) \longrightarrow P(A)$ such that $\lambda(T) = \emptyset$, for any T in P(A). It is clear that λ is BCK- homomorphism and $P(B)$ is a prime BCK- submodule of $P(A)$, but $\lambda(P(B)) = \emptyset$ is not a prime BCK-submodule of $P(A)$, because if $x = \{1\}$ and $y = \{2\}$, then x and y are subsets of A and $x \cap y = \emptyset$ whereas $x \neq \emptyset$ and $y \cap P(A) = \{2\} \neq \emptyset$.*

Let X be a lower semilattice BCK- algebra. Then $N(X)$ will denote the intersection of all prime ideals of X.

Theorem 10. *Let P be a prime ideal of a lower semilattice X containing I. Then $\frac{P}{I}$ is a prime ideal of BCK- algebra $\frac{X}{I}$.*

Proof. First we show $\frac{P}{I} \neq \frac{X}{I}$. If $\frac{P}{I} = \frac{X}{I}$, then $X = P$, because $x \in X$, implies that $C_x \in \frac{X}{I} = \frac{P}{I}$ i.e. $C_x = C_{p_1}$, for some $p_1 \in P$. So $x * p_1 \in I \subseteq P$. Hence $x \in P$. Therefore $X = P$, this is a contradiction. Now let $(C_x) \wedge (C_y) \in \frac{P}{I}$. Then $C_{x\wedge y} \in \frac{P}{I}$. It is easy to see that $x \wedge y \in P$. By primitivity P, we get that $C_x \in \frac{P}{I}$ or $C_y \in \frac{P}{I}$. Therefore $\frac{P}{I}$ is a prime ideal of BCK- algebra $\frac{X}{I}$.

Theorem 11. *Let M be a left BCK- module over X such that* $\hom(M, \frac{X}{N(X)}) \neq 0$. *Then M contains a prime BCK- submodule.*

Proof. Since $\hom(M, \frac{X}{N(X)}) \neq 0$, there exists a BCK- homomorphism υ such that $\upsilon(m_0) \neq N(X)$, for some $m_0 \in M$. In the other hand there exists $x_0 \in X$ such that $\upsilon(m_0) = C_{x_0}$ and $C_{x_0} \neq C_0$, hence $x_0 \notin N(X)$. i.e. there exists a prime ideal P_0 of X such that $x_0 \notin P_0$.
Since $C_{x_0} \notin C_{P_0}$, we get that $\upsilon(M) \nsubseteq C_{P_0}$. By theorem 10 C_{P_0} is a prime ideal of $\frac{X}{N(X)}$. So by Theorem 9 $\upsilon^{-1}(C_{P_0})$ is a prime BCK- submodule of M.

Theorem 12. *Let A be an ideal of X and M be a left BCK- module over X. Then there exists a proper BCK- submodule N of M such that* $A = (N : M)$ *if and only if* $A \cdot M \neq M$ *and* $A = (A \cdot M : M)$.

Proof. The sufficiency is clear.
Conversely, suppose that $A = (N : M)$, for some proper BCK- submodule N of M, since $A \cdot M \subseteq N$, we have $A \cdot M \neq M$.
Moreover clearly $A \subseteq (A \cdot M : M)$, it is sufficient to show that $(A \cdot M : M) \subseteq A$. Let $x \in (A \cdot M : M)$. Then $x \cdot M \subseteq A \cdot M$, so $x \cdot M \subseteq N$ i.e. $x \in (N : M)$.

Let M be a left BCK- module over lower semilattice X and P be a prime ideal of X. Then we shall denote by M(P) the following subset of M:
$M(P) = \{m \in M \mid A \cdot m \subseteq P \cdot M$, for some ideal $A \nsubseteq P\}$.
It is clear that M(P) is a BCK- submodule of M and $P \cdot M \subseteq M(P)$.
Note the following fact about M(P).

Theorem 13. *Let P be a prime ideal of a lower semilattice X and M be a left BCK- module over X such that there exists a prime BCK- submodule K of M with* $(K : M) = P$. *Then* $M(P) \subseteq K$.

Proof. Let $m \in M(P)$. Then there is an ideal A of X such that $A \nsubseteq P$ and $A \cdot m \subseteq P \cdot M$.
Since $P \cdot M \subseteq K$, then we have $A \cdot m \subseteq K$ and $A \nsubseteq P$, so $a_1 \notin P$, for some $a_1 \in A$. In the other hand, $A \cdot m \subseteq K$, hence $a_1 \cdot m \in K$. By primitivity K , we have $m \in K$ or $a_1 \cdot M \subseteq K$. If $a_1 \cdot M \subseteq K$, then we have $a_1 \in (K : M) = P$, therefore $a_1 \in P$. This is a contradiction. So $m \in K$. The proof is complete.

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