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CONCAVITY OF RUSCHEWEYH DIFFERENTIAL OPERATOR

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Abstract. In this article we introduce a class of concave analytic functions $D^nC_0(\alpha)$ defined by Ruscheweyh differential operator. We study some properties such as coefficient inequalities and distortion theorems for this class.

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1. Introduction

The study of operators plays an important role in mathematics especially in the geometric function theory. Recently interest in this direction has been increasing as it permits detailed investigations of problems with physical applications. For example, the differential operator is linked between function theory and mathematical physics. Sălăgean [19], Noor and Noor [16, 17], Noor [15] and many others , for example, [9, 13], defined new operators and studied various classes of analytic and univalent functions which generalized a number of previously known classes and at times discovering new classes of analytic functions.

In 1975 Ruscheweyh [18] defined the differential operator D^n of the class of analytic functions by using the technique of convolution. Many authors have used the Ruscheweyh operator to define and investigate the properties of certain known and new classes of analytic functions. We mention some of them in recent years.

In 2010, Lupas [1] defined a new operator using the Sălăgean and Ruscheweyh operators, and studied some differential subordinations regarding the new operator.

Also, Lupas [2] studied a new operator $DI^{\alpha}_{n,\lambda,l}$ using the multiplier transformation and Ruscheweyh derivative given by

$$DI^{\alpha}_{n,\lambda,I} : \mathcal{A} \to \mathcal{A},\ DI^{\alpha}_{n,\lambda,I}f(z) = (1-\alpha)D^nf(z) + \alpha I(n,\lambda,l)f(z),\ z \in \mathbb{D},$$

where $D^nf(z)$ denote the Ruscheweyh derivative and $I(n,\lambda,I)f(z)$ is the multiplier transformation. Several differential subordinations were established regarding the

operator $DI^{\alpha}_{n,\lambda,l}$.
Najafzadeh [14] introduced a new subclass of holomorphic univalent functions with negative and fixed finitely coefficient based on Sălăgean and Ruscheweyh differential operators defined by

$$\Omega^n_\lambda : \mathcal{A} \to \mathcal{A},\ \Omega^n_\lambda f(z) = (1-\lambda)S^n f(z) + \lambda D^n f(z),\ z \in \mathbb{D},\ n \in \mathbb{N} \cup \{0\}\ and\ \lambda \geq 0.$$

In 2011, again Lupas [3] derived a new operator using the generalized Sălăgean and Ruscheweyh operators, given by

$$DR^m_\lambda : \mathcal{A} \to \mathcal{A},\ DR^m_\lambda f(z) = (D^m_\lambda * R^m)f(z),$$

where DR^m_λ is the Hadamard product of the generalized Sălăgean operator D^m_λ and Ruscheweyh operator R^m.

In 2012, Lupas [4] established several strong differential superordinations regarding the new operator SD^m defined by convolution product of the extended Sălăgean operator and Ruscheweyh derivative,

$$SD^m : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta,\ SD^m f(z,\zeta) = (S^m * D^m)f(z,\zeta),\ z \in \mathbb{D}\ and\ \zeta \in \bar{\mathbb{D}},$$

where $D^m f(z,\zeta)$ denote the extended Ruscheweyh derviative and $S^m f(z,\zeta)$ is the extended Sălăgean operator.

Later Swamy [20] gave a new operator $DI^m_{\alpha,\beta,\lambda}$ defined by

$$DI^m_{\alpha,\beta,\lambda} f(z) = (1-\lambda)D^m f(z) + \lambda I^m_{\alpha,\beta} f(z),\ \lambda \geq 0,$$

where $D^m f(z)$ is the Ruscheweyh operator.
Moreover, Lupas [5] studied the operator defined by using the Ruscheweyh derivative $D^n f(z)$ and the Sălăgean operator $S^n f(z)$, denoted by

$$L^n_\alpha : \mathcal{A} \to \mathcal{A}, L^n_\alpha f(z) = (1-\alpha)D^n f(z) + \alpha S^n f(z),\ z \in \mathbb{D}.$$

2. Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{D}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ...$ Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{D}. \tag{1}$$

Let $n \in \mathbb{N}_0 = 0, 1, 2, 3, ...$ The Ruscheweyh derivative of n^{th} of order f, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^n}{n!}, \; n \in \mathbb{N}_0.$$

Ruscheweyh determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} C(n,k) a_k z^k, \tag{2}$$

where $C(n,k) = \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n+1)}, k \geq 2, n \geq 0$ such that

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z f'(z)$$

$$2D^2 f(z) = z(D' f(z))' + D' f(z)$$

$$(n+1) D^{n+1} f(z) = z(D^n f(z))' + n D^n f(z).$$

A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$, if f satisfies the following conditions:

(i) f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition, it satisfies $f(1) = \infty$.

(ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.

(iii) The opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha, \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to concave univalent functions and for a detailed discussion about concave functions we refer to [6],[7],[10] and the references therein.
We recall the analytic characterization for the functions in $C_0(\alpha), \alpha \in (1, 2] : f \in C_0(\alpha)$ if and only if $ReP(z) > 0, z \in \mathbb{D}$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right]$$

In [8] they used this characterization.

Now in the following definition, we define new subclasses of concave analytic functions containing Ruscheweyh differential operator.

Definition 1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in D^nC_0(\alpha)$ if and only if*

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{[D^nf(z)]''}{[D^nf(z)]'}\right]>0,$$

$\alpha \in (1,2]$, $n \geq 0$ and $z \in \mathbb{D}$. Where $D^nf(z)$ is given by (2).

Remark 1. *When $n = 0$, we get the class of concave univalent functions.*

The object of the present paper is to investigate some new properties of this class.

3. General properties of $D^nC_0(\alpha)$

In this part, we study the coefficient estimates for functions of the form (1) in the class $D^nC_0(\alpha)$.

Theorem 1. *Let $f(z) \in \mathcal{A}$. If for $\alpha \in (1,2]$, $n \geq 0$ and*

$$\sum_{k=2}^{\infty}|a_k|C(n,\,k)(k^2+\alpha k)<\alpha-1 \tag{3}$$

then $f(z) \in D^nC_0(\alpha)$. The result (3) is sharp.

Proof. We want to prove that

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{zg'(z)}{g(z)}\right]>0$$

By using the fact $Re\, w > 0 \longleftrightarrow |1+w| > |1-w|$. It is enough to show that $|w| < 1$. Let $w = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{zg'(z)}{g(z)}\right]$, $g(z) = z(D^nf(z))'$ and $D^nf(z) = z+\sum_{k=0}^{\infty}C(n,\,k)a_kz^k$. So we have

$$g(z)=z\left(1+\sum_{k=2}^{\infty}ka_kC(n,\,k)z^{k-1}\right)$$

$$g'(z)=1+\sum_{k=2}^{\infty}k^2a_kC(n,\,k)z^{k-1}$$

We Know that

$$\begin{aligned}
&\left|\frac{\alpha+1}{2}\frac{1+z}{1-z}+\frac{1+\sum_{k=2}^{\infty}k^2a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right| \\
&\le \frac{\alpha+1}{2}\left|\frac{1+z}{1-z}\right|+\left|\frac{1+\sum_{k=2}^{\infty}k^2a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right| \\
&\le \frac{\alpha+1}{2}\left|\frac{1+z}{1-z}\right|+\frac{1+\sum_{k=2}^{\infty}k^2|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}} \\
&\le \frac{\alpha+1}{2}\left(\left|\frac{1+z}{1-z}\right|+\frac{1+\sum_{k=2}^{\infty}k^2|a_k||z^{k-1}|}{1-\sum_{k=2}^{\infty}k|a_k||z^{k-1}|}\right)
\end{aligned}$$

By the assumption on (3) and $z \to -1$, we have

$$\frac{1+\sum_{k=2}^{\infty}k^2|a_k|}{1-\sum_{k=2}^{\infty}k|a_k|} \le \frac{\alpha-1}{2}.$$

We obtain

$$\left|\frac{\alpha+1}{2}\frac{1+z}{1-z}+\frac{1+\sum_{k=2}^{\infty}k^2a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right| \le \frac{\alpha-1}{2}.$$

This implies

$$\left|\frac{\alpha+1}{2}\frac{1+z}{1-z}\right|-\left|\frac{1+\sum_{k=2}^{\infty}k^2a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right| < \frac{\alpha-1}{2}$$

$$\left|\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{1+\sum_{k=2}^{\infty}k^2a_kz^{k-1}}{1+\sum_{k=2}^{\infty}ka_kz^{k-1}}\right| < \frac{\alpha-1}{2}.$$

So we have

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{zg'(z)}{g(z)}\right] > 0.$$

Finally the result is sharp with the extremal function f given by

$$f(z) = z+\sum_{k=2}^{\infty}\frac{\alpha-1}{(k^2+\alpha k)C(n,\, k)}z^k.$$

Corollary 2. *Let the assumption of Theorem 1 be satisfied. Then*

$$|a_k| \le \frac{\alpha-1}{(k^2+\alpha k)C(n,\, k)},\ \forall k \ge 2,\ \alpha \in (1, 2]..$$

Corollary 3. *Let the assumption of Theorem 1 be satisfied. Then for* $n = 0$

$$|a_k| \leq \frac{\alpha - 1}{k^2 + \alpha k}, \; \forall k \geq 2, \; \alpha \in (1, 2].$$

Next, by using inequality (3) the following theorems gives distortion bounds for functions contained in the class $D^n C_0(\alpha)$*.*

Theorem 4. *Let the assumption of Theorem 1 be satisfied. Then for* $z \in \mathbb{D}$ *and* $\alpha \in (1, 2]$

$$|D^n f(z)| \geq |z| - \frac{\alpha - 1}{2(\alpha + 2)} |z|^2$$

and

$$|D^n f(z)| \leq |z| + \frac{\alpha - 1}{2(\alpha + 2)} |z|^2.$$

.

Proof. From Theorem 1, we have

$$2(\alpha + 2) \sum_{k=2}^{\infty} C(n, k)|a_k| \leq \sum_{k=2}^{\infty} C(n, k)|a_k|(k^2 + \alpha k) \leq \alpha - 1$$

That is

$$\sum_{k=2}^{\infty} C(n, k)|a_k| \leq \frac{\alpha - 1}{2(\alpha + 2)}$$

According to (2) we obtain

$$\begin{aligned} |D^n f(z)| &\leq |z| + \sum_{k=2}^{\infty} C(n, k)|a_k||z|^k \\ &\leq |z| + \sum_{k=2}^{\infty} C(n, k)|a_k||z|^2 \\ &\leq |z| + \frac{\alpha - 1}{2(\alpha + 2)} |z|^2 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}|D^n f(z)| &= \left| z + \sum_{k=2}^{\infty} C(n,\, k) a_k z^k \right| \\ &\geq |z| - \sum_{k=2}^{\infty} C(n,\, k)|a_k||z|^k \\ &\geq |z| - \sum_{k=2}^{\infty} C(n,\, k)|a_k||z|^2 \\ &\geq |z| - \frac{\alpha - 1}{2(\alpha + 2)}|z|^2.\end{aligned}$$

This completes the proof.

Also the next theorem provides distortion theorem.

Theorem 5. *Let the assumption of Theorem 1 be satisfied. Then for* $z \in \mathbb{D}$ *and* $\alpha \in (1, 2]$

$$|f(z)| \geq |z| - \frac{(\alpha - 1)\Gamma(n + 1)}{2(3 - \alpha)\Gamma(n + 2)}|z|^2$$

and

$$|f(z)| \leq |z| + \frac{(\alpha - 1)\Gamma(n + 1)}{2(3 - \alpha)\Gamma(n + 2)}|z|^2$$

.

Proof. According to the Theorem 1 we get that

$$2(\alpha + 2)\frac{\Gamma(n + 2)}{\Gamma(n + 1)} \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} C(n,\, k)|a_k|(k^2 + \alpha k) \leq \alpha - 1$$

Thus we get

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(\alpha - 1)\Gamma(n + 1)}{2(\alpha + 2)\Gamma(n + 2)}$$

Next from (1), we have

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \\
&\leq |z| + \sum_{k=2}^{\infty} |a_k||z|^k \\
&\leq |z| + \sum_{k=2}^{\infty} |a_k||z|^2 \\
&\leq |z| + \frac{(\alpha-1)\Gamma(n+1)}{2(\alpha+2)\Gamma(n+2)}|z|^2
\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \\
&\geq |z| - \sum_{k=2}^{\infty} |a_k||z|^2 \\
&\geq |z| - \frac{(\alpha-1)\Gamma(n+1)}{2(\alpha+2)\Gamma(n+2)}|z|^2
\end{aligned}$$

This completes the proof.

4. Conclusion

The class of concave analytic functions $D^nC_0(\alpha)$ defined by Ruscheweyh differential operator are introduced. Characterization and other properties of this class are studied.

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