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SOLUTION OF BRATU-TYPE EQUATION VIA SPLINE METHOD

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Abstract. In this paper, non-polynomial spline method is applied for solving Bratu's problem. The convergence analysis of the presented method is discussed. The method is illustrated with two numerical examples and the results show that the method converges rapidly and approximates the exact solution very accurately.

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1. Introduction

Consider the Liouville-Bratu-Gelfand equation [1]-[2]

$$\begin{cases} \Delta u(x) + \lambda e^{u(x)} = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$, and Ω is a bounded domain. We consider the Bratu's boundary value problem in one-dimensional planar coordinates [2]-[4] of the form

$$\begin{aligned} u''(x) + \lambda e^{u(x)} = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{2}$$

The Bratu's problem is widely used in science and engineering to describe complicated physical and chemical models. For example, Bratu's problem [2]-[6] is used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of the thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [7]-[11].

Several numerical methods for approximating the solution of Bratu's problem are known. Laplace transform decomposition numerical algorithm is used for solving Bratu's problem [12]. The Perturbation-iteration algorithm [13], applied to Bratu-type equations.Mohsen et al. [14] introduced new smoother to enhance multigrid-based methods for Bratu problem. One-point pseudospectral collocation method has been used for the solution of the one-dimensional Bratu equation [15]. The main purpose of the present paper is to use non-polynomial cubic spline method [16]-[17] for the numerical solution of nonlinear boundary value problem (2). The method consists of reducing the problem to a set of algebraic equations.

The outline of the paper is as follows. First, in Section 2 we introduce non-polynomial spline method and describe the basic formulation of spline approximation required for our subsequent development. In section 3, the convergence analysis of the method has been discussed. Finally we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering two numerical examples in Section 4.

2. Parametric Cubic spline method

We obtain an approximate solution of (2) using non-polynomial cubic spline functions. Let $S_k(\Delta)$ denote the set of piecewise polynomials of degree k on subinterval $I_i = [x_i, x_{i+1}]$ of partition $\Delta = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b] \subset R$. Consider the grid points x_i on the interval $[a, b]$ as follows:

$$a = x_0 < x_1 < x_2 < \ldots, x_{n-1} < x_n = b, \tag{3}$$

$$x_i = a + (i + \frac{1}{2})h, \quad i = 0, 1, 2, \ldots, n-1, \tag{4}$$

$$h = \frac{b-a}{n}, \tag{5}$$

where n is a positive integer. Let $u(x)$ be the exact solution of the Eq.(2) and $S_i(x)$ be an approximation to $u_i = u(x_i)$ obtained by the segment $P_i(x)$. Each non-polynomial spline segment $P_i(x)$ has the form:

$$P_i(x) = a_{i+\frac{1}{2}} \sin k(x - x_{i+\frac{1}{2}}) + b_{i+\frac{1}{2}} \cos k(x - x_{i+\frac{1}{2}}) + c_{i+\frac{1}{2}}(x - x_{i+\frac{1}{2}}) + d_{i+\frac{1}{2}}, \quad i = 0, 1, \ldots, n-1, \tag{6}$$

where $a_{i+\frac{1}{2}}$, $b_{i+\frac{1}{2}}$, $c_{i+\frac{1}{2}}$ and $d_{i+\frac{1}{2}}$ are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and Eq. (6) reduce to cubic polynomial spline function in $[a, b]$ when $k \to 0$. For convenience consider the following relations:

$$\begin{aligned} P_i(x_{i+\frac{1}{2}}) &= u_{i+\frac{1}{2}}, & P_i(x_{i+\frac{3}{2}}) &= u_{i+\frac{3}{2}}, \\ P_i'(x_{i+\frac{1}{2}}) &= D_{i+\frac{1}{2}}, & P_i'(x_{i+\frac{3}{2}}) &= D_{i+\frac{3}{2}}, \\ P_i''(x_{i+\frac{1}{2}}) &= M_{i+\frac{1}{2}}, & P_i''(x_{i+\frac{3}{2}}) &= M_{i+\frac{3}{2}}. \end{aligned} \tag{7}$$

We can obtain the values of $a_{i+\frac{1}{2}}$, $b_{i+\frac{1}{2}}$, $c_{i+\frac{1}{2}}$ and $d_{i+\frac{1}{2}}$ via a straightforward calculation as follows:

$$a_{i+\frac{1}{2}} = \frac{M_{i+\frac{1}{2}}\cos\theta - M_{i+\frac{3}{2}}}{k^2\sin\theta}, \quad b_{i+\frac{1}{2}} = -h^2\frac{M_{i+\frac{1}{2}}}{\theta^2}, \tag{8}$$

$$\begin{aligned} c_{i+\frac{1}{2}} &= \frac{D_{i+1}+D_{i+\frac{1}{2}}}{2} - \frac{\big(M_{i+\frac{1}{2}}\cos\theta - M_{i+\frac{3}{2}}\big)\big(1+\cos\theta\big)}{2k\sin\theta} \\ &\quad - \frac{M_{i+\frac{1}{2}}\sin\frac{\theta}{2}}{2k}, \end{aligned} \tag{9}$$

$$d_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} + \frac{M_{i+\frac{1}{2}}}{k^2}, \tag{10}$$

where $\theta = kh$ and $i = 0, 1, \ldots, n-1$.

Using the continuity conditions $P_{i-1}^{(n)}(x_{i+\frac{1}{2}}) = P_i^{(n)}(x_{i+\frac{1}{2}})$, $n = 0, 1$, we get the following relations for $i = 0, 1, \ldots, n-1$:

$$\begin{aligned} \frac{h}{2}\big(D_i + D_{i-\frac{1}{2}}\big) &= u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}\Big(\frac{1}{k^2} - \frac{h\big(\cos\frac{\theta}{2}+1\big)}{2k\sin\theta}\Big) \\ &\quad + M_{i-\frac{1}{2}}\Big(\frac{h\cos\theta\cos\frac{\theta}{2} + h\cos\theta + h\sin\theta\sin\frac{\theta}{2}}{2k\sin\theta} - \frac{1}{k^2}\Big), \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\frac{1}{2}\big(D_i + D_{i-\frac{1}{2}} - D_{i+1} - D_{i+\frac{1}{2}}\big) = \\ &\quad M_{i+\frac{1}{2}}\Big(\frac{\cos\theta}{k\sin\theta} - \frac{\sin\frac{\theta}{2}}{2k} - \frac{\cos\theta\big(1+\cos\frac{\theta}{2}\big)}{2k\sin\theta} + \frac{1}{k\tan\theta} - \frac{1+\cos\frac{\theta}{2}}{2k\sin\theta}\Big) \\ &\quad + M_{i-\frac{1}{2}}\Big(\frac{-\cos\theta}{k\tan\theta} - \frac{\sin\theta}{k} + \frac{\cos\theta\big(1+\cos\frac{\theta}{2}\big)}{2k\sin\theta} + \frac{\sin\frac{\theta}{2}}{2k}\Big) \\ &\quad + M_{i+\frac{3}{2}}\Big(-\frac{1}{k\sin\theta} + \frac{1+\cos\frac{\theta}{2}}{2k\sin\theta}\Big). \end{aligned} \tag{12}$$

By reducing the indices of Eqs. (11) and (12) by one, we get the following equations:

$$\frac{1}{2}\big(D_{i-1}+D_{i-\frac{3}{2}}\big)=\frac{u_{i-\frac{1}{2}}-u_{i-\frac{3}{2}}}{h}+M_{i-\frac{1}{2}}\Big(\frac{1}{hk^2}-\frac{\cos\frac{\theta}{2}+1}{2k\sin\theta}\Big)$$
$$+\,M_{i-\frac{3}{2}}\Big(\frac{\cos\frac{\theta}{2}+\cos\theta}{2k\sin\theta}-\frac{1}{hk^2}\Big), \qquad (13)$$

and also

$$\frac{1}{2}\big(D_{i-1}+D_{i-\frac{3}{2}}-D_i-D_{i-\frac{1}{2}}\big)=$$
$$M_{i-\frac{1}{2}}\Big(\frac{\cos\theta}{k\sin\theta}-\frac{\sin\frac{\theta}{2}}{2k}-\frac{\cos\theta\big(1+\cos\frac{\theta}{2}\big)}{2k\sin\theta}+\frac{1}{k\tan\theta}-\frac{1+\cos\frac{\theta}{2}}{2k\sin\theta}\Big)$$
$$+M_{i-\frac{3}{2}}\Big(\frac{-\cos\theta}{k\tan\theta}-\frac{\sin\theta}{k}+\frac{\cos\theta\big(1+\cos\frac{\theta}{2}\big)}{2k\sin\theta}+\frac{\sin\frac{\theta}{2}}{2k}\Big)$$
$$+M_{i+\frac{1}{2}}\Big(-\frac{1}{k\sin\theta}+\frac{1+\cos\frac{\theta}{2}}{2k\sin\theta}\Big). \qquad (14)$$

$D_{i+j},\ j=-\frac{3}{2},-1,0,\frac{1}{2}$ are eliminated from Eq. (14) by using Eq. (13). Now we get the following scheme:

$$\frac{1}{h}\Big(2u_{i-\frac{1}{2}}-u_{i+\frac{1}{2}}-u_{i-\frac{3}{2}}\Big)=M_{i+\frac{1}{2}}\Big(\frac{1}{hk^2}-\frac{1}{k\sin\theta}\Big)+$$
$$M_{i-\frac{3}{2}}\Big(\frac{1}{hk^2}-\frac{1}{k\sin\theta}\Big)+M_{i-\frac{1}{2}}\Big(\frac{-2}{hk^2}+\frac{2\cos\theta}{k\sin\theta}\Big). \qquad (15)$$

Consider

$$\alpha=\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2},\qquad \beta=\frac{1}{\theta^2}-\frac{\cos\theta}{\theta\sin\theta}, \qquad (16)$$

then, we can write:

$$u_{i+\frac{1}{2}}-2u_{i-\frac{1}{2}}+u_{i-\frac{3}{2}}=h^2\Big[\alpha M_{i-\frac{3}{2}}+2\beta M_{i-\frac{1}{2}}+\alpha M_{i+\frac{1}{2}}\Big],\ \ i=2,3,\dots,n-1, \qquad (17)$$

For a numerical solution of the Bratu's problem (2), the interval $[0,1]$ is divided into a set of grid points with step size h. Setting $x=x_i=x_0+(i-\frac{1}{2})h$, in Eq. (22), we obtain

$$u''(x_i)=-\lambda e^{u(x_i)}, \qquad (18)$$

by using the assumption $S''_\Delta(x_i)=M_i$ in (17) we have

$$M_i=-\lambda e^{u(x_i)}. \qquad (19)$$

Replacing $M_{i+j},\ j=-\frac{3}{2},-\frac{1}{2},\frac{1}{2}$ as Eq. (19) in Eq.(17), we get

$$\big(\lambda\alpha h^2 e^{u_{i-\frac{3}{2}}}+u_{i-\frac{3}{2}}\big)+\big(2\lambda\beta h^2 e^{u_{i-\frac{1}{2}}}-2u_{i-\frac{1}{2}}\big)+\big(\lambda\alpha h^2 e^{u_{i+\frac{1}{2}}}+u_{i+\frac{1}{2}}\big)=0,\quad i=2,3,\ldots,n-1. \tag{20}$$

Using Taylor's series for Eq. (20), we can obtain local truncation error as follows:

$$t_i=h^2\big(2\alpha+2\beta-1\big)u_i''+h^3\big(-\alpha-\beta+\frac{1}{2}\big)u_i^{(3)}+h^4\big(-\frac{5}{24}+\frac{5}{4}\alpha+\frac{1}{4}\beta\big)u_i^{(4)}+h^5\big(\frac{1}{16}-\frac{13}{24}\alpha-\frac{1}{24}\beta\big)u_i^{(5)}+h^6\big(\frac{41}{192}\alpha+\frac{1}{192}\beta-\frac{91}{5760}\big)u_i^{(6)}+O\big(h^7\big),\quad i=2,3,\ldots,n-1. \tag{21}$$

The nonlinear system (20) consists of $(n-2)$ equation with n unknowns $u_{i-1/2},\ i=1,\ldots,n$. To obtain unique solution, we need two equations. For this purpose, we can use the following equations that are found by using method of undetermined coefficients.

$$2u_0-3u_{\frac{1}{2}}+u_{\frac{3}{2}}=h^2\big(\frac{-1}{120}M_0+\frac{5}{8}M_{\frac{1}{2}}+\frac{7}{48}M_{\frac{3}{2}}-\frac{1}{80}M_{\frac{5}{2}}\big),\quad i=1, \tag{22}$$

$$u_{n-\frac{3}{2}}-3u_{n-\frac{1}{2}}+2u_n=h^2\big(\frac{-1}{120}M_n+\frac{5}{8}M_{n-\frac{1}{2}}+\frac{7}{48}M_{n-\frac{3}{2}}-\frac{1}{80}M_{n-\frac{5}{2}}\big),\quad i=n, \tag{23}$$

where $M_j=-\lambda e^{u(x_j)}$. The nonlinear systems (20), (22) and (23) consist of n equations with n unknowns $u_{i-1/2},\ i=1,\ldots,n$. Solving this nonlinear system by $Newton's$ method, we can obtain an approximation to the solution of (2). The local truncation errors associated with the scheme (20), (22) and (23) can be obtained as follows:

$$t_i=\begin{cases}\frac{19}{5120}h^6u_0^{(6)}+O(h^7), & i=1,\\ \frac{1}{240}h^6u_i^{(6)}+O(h^7), & i=2,3,\ldots,n-1,\\ \frac{19}{5120}h^6u_n^{(6)}+O(h^7), & i=n,\end{cases}$$

with $\alpha=\frac{1}{12},\ \beta=\frac{5}{12}$.

3. Convergence analysis

Now we discuss the convergence of the non-polynomial spline method for the Bratu's problem (2). We consider the Eqs. (20), (22) and (23) and then rewrite these equations in the matrix form which is the nonlinear system as:

$$AU+\lambda h^2BG=0, \tag{24}$$

where $U = \left(u_{\frac{1}{2}}, \ldots, u_{n-\frac{1}{2}}\right)^T$. Also $A = \left[a_{i,j}\right]$, $B = \left[b_{i,j}\right]$ are tridiagonal matrices of order $n \times n$ and define as follows:

$$a_{i,j} = \begin{cases} 3, & i = j = 1, n, \\ 2, & i = j,\ i = 2, \ldots, n-1, \\ -1, & |i-j| = 1, \\ 0, & o.w, \end{cases}$$

$$b_{i,j} = \begin{cases} -\frac{15}{24}, & i = j = 1, n, \\ -\frac{3}{24}, & i = 1, n,\ j = 2, n-1, \\ -2\beta, & i = j,\ i = 2, \ldots, n-1, \\ -\alpha, & |i-j| = 1, i \neq 1, n, \\ 0, & o.w, \end{cases}$$

and $G = diag\left(e^{u_{i-\frac{1}{2}}}\right),\ i = 1, 2, \ldots, n.$

Theorem 1. *Let M be a matrix such that $||M|| < 1$, and let I denote the unit matrix. Then $(I+M)^{-1}$ exists, and*

$$||(I+M)^{-1}|| < \frac{1}{1-||M||}. \tag{25}$$

Proof. By applying Theorem 1.7.7, Ref.[18], we can conclude the Eq. (25).

From Eq. (24) we can write:

$$C = AU + \lambda h^2 BG. \tag{26}$$

The inverse of A exists and bounded as follows [17]:

$$||A^{-1}|| \leq \frac{(b-a)^2 + h^2}{8h^2}. \tag{27}$$

Theorem 2. *If $||G|| < \frac{8}{\lambda\left((b-a)^2+h^2\right)}$ then the inverse of C exists.*

Proof. Considering Eq.(25) we can write:

$$C = A(I + \lambda h^2 A^{-1} BG), \tag{28}$$

from Eq.(27), we conclude that A^{-1} exists. Now, we need the existence of $\big(I + \lambda h^2 A^{-1} BG\big)^{-1}$. According to Theorem 1 is sufficient, we show that $||\lambda h^2 A^{-1} BG|| < 1$. Having used Eq.(27) and also $||B|| = 1$, we obtain:

$$\begin{aligned} ||\lambda h^2 A^{-1} BG|| &\leq \lambda h^2 ||A^{-1}|| ||B|| ||G|| \\ &< \frac{\lambda(b-a)^2 + h^2}{8} ||G||. \end{aligned} \tag{29}$$

Considering assumption $||G|| < \frac{8}{\lambda\big((b-a)^2+h^2\big)}$, we have:

$$||\lambda h^2 A^{-1} BG|| < 1. \tag{30}$$

Therefore, by using Theorem 1 and Eqs. (27) and (30) we conclude the existence of C^{-1}. □

We can also obtain a bound on the errors $E = U - U_n$ in the maximum norm, where $U = \big(u(x_{\frac{1}{2}}), \ldots, u(x_{n-\frac{1}{2}})\big)$ is the exact solution and $U_n = \big(u_{\frac{1}{2}}, \ldots, u_{n-\frac{1}{2}}\big)$ is the approximate solution of Bratu's problem (2). From Theorem 2, we can derive a bound on $||E||$ and show that the non-polynomial spline method converges at rate of $O\big(h^4\big)$.

Theorem 3. *Let T be the vector of local truncation error and $CE = T$, then*

$$||E|| \cong O(h^4), \ \big(\text{when } \alpha = \frac{1}{12}, \ \ \beta = \frac{5}{12}\big). \tag{31}$$

Proof. . By using Theorem 2 and $CE = T$, we can write:

$$E = C^{-1}T = \big(I + \lambda h^2 A^{-1} BG\big)^{-1} A^{-1} T, \tag{32}$$

therefore, we get

$$||E|| \leq ||\big(I + \lambda h^2 A^{-1} BG\big)^{-1}||||A^{-1}||||T|| \tag{33}$$

Having used Eq. (19) and applied Theorem 1, we obtain

$$||\big(I + \lambda h^2 A^{-1} BG\big)^{-1}|| \leq \frac{1}{1 - ||\lambda h^2 A^{-1} BG||}. \tag{34}$$

By applying Eqs. (27) and (32) and also $||B|| = 1$, we have:

$$||E|| \leq \frac{(b-a)^2}{h^2\big(8 - \lambda G(b-a)^2\big)}||T||. \tag{35}$$

For $||T||$ when $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ then $||T|| \leq \frac{h^6}{240}M_6$, where $M_6 = max|u^{(6)}(x)|$.

Therefore from Eq. (34), we conclude that

$$||E|| \leq \frac{(b-a)^2}{h^2\big(8 - \lambda G(b-a)^2\big)}\frac{h^6}{240}M_6 \cong O\big(h^4\big), \tag{36}$$

this completes the proof of Theorem 3. □

Theorem 4. *Let T be the vector of local truncation error and $CE = T$, then*

$$||E|| \cong O(h^4), \ \big(when\ \alpha = \frac{1}{12}, \ \ \beta = \frac{5}{12}\big). \tag{37}$$

Proof. By using Theorem 2 and $CE = T$, we can write:

$$E = C^{-1}T = \big(I + \lambda h^2 A^{-1}BG\big)^{-1}A^{-1}T, \tag{38}$$

therefore, we get

$$||E|| \leq ||\big(I + \lambda h^2 A^{-1}BG\big)^{-1}||||A^{-1}||||T|| \tag{39}$$

Having used Eq. (19) and applied Theorem 1, we obtain

$$||\big(I + \lambda h^2 A^{-1}BG\big)^{-1}|| \leq \frac{1}{1 - ||\lambda h^2 A^{-1}BG||}. \tag{40}$$

By applying Eqs. (27) and (32) and also $||B|| = 1$, we have:

$$||E|| \leq \frac{(b-a)^2}{h^2\big(8 - \lambda G(b-a)^2\big)}||T||. \tag{41}$$

For $||T||$ when $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ then $||T|| \leq \frac{h^6}{240}M_6$, where $M_6 = max|u^{(6)}(x)|$.
Therefore from Eq. (34), we conclude that

$$||E|| \leq \frac{(b-a)^2}{h^2\big(8 - \lambda G(b-a)^2\big)}\frac{h^6}{240}M_6 \cong O\big(h^4\big), \tag{42}$$

this completes the proof of Theorem 3.

4. Numerical illustrations

In order to illustrate the performance of the non-polynomial spline method for the Bratu equation (2) and justify the accuracy and efficiency of the method, we consider the following examples. The example have been solved by presented method with different values of λ. we take $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 10$. The errors are reported on the set of uniform grid points

$$S = \{x_1, x_2, \dots, x_n\},$$
$$x_i = x_0 + (i - \frac{1}{2})h, \quad i = 1, 2, \dots, n, \ \ h = \frac{b-a}{n}. \tag{43}$$

The absolute error on the uniform grid points S is

$$|u(x_j) - u_j|, \quad 1 \leq j \leq n, \tag{44}$$

where $u(x_j)$ is the exact solution of the given example, and u_j is the computed solution by the non-polynomial cubic spline method. The exact solution of the equation (2) is given in [1] and [2] as:

$$u(x) = -2 \ln \Big[\frac{\cosh\left((x - \frac{1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)} \Big] \tag{45}$$

where θ satisfies

$$\theta = \sqrt{2\lambda} \cosh\big(\frac{\theta}{4}\big). \tag{46}$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c} \sinh\big(\frac{\theta_c}{4}\big). \tag{47}$$

It was evaluated in [1] and [2] that the critical value λ_c is given by $\lambda_c = 3.513830719$. The absolute errors in computed solutions are tabulated in Table 1-2.

Example 1. *We first consider the Bratu-type model*

$$\begin{aligned} &u''(x) + \lambda e^{u(x)} = 0, \quad 0 < x < 1, \\ &u(0) = u(1) = 0, \end{aligned} \tag{48}$$

where $\lambda = 1$. The numerical results for Example 1 are tabulated in Table 1.

Example 2. *Consider the second case for Bratu equation as follows when $\lambda = 2$.*

$$\begin{aligned} &u''(x) + 2e^{u(x)} = 0, \quad 0 < x < 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{49}$$

In Table 2, the exact solution for the case $\lambda = 2$ is compared with the numerical solution obtained by the parametric spline method.

Table 1: $\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 10$.

x	*Exact solution*	*Numerical solution*	*Error*
0.05	0.026206378688	0.026205652306	7.26382×10^{-7}
0.15	0.070859894772	0.070859219050	6.75722×10^{-7}
0.25	0.104787310536	0.104786680206	6.30330×10^{-7}
0.35	0.127619498329	0.127618902166	5.96163×10^{-7}
0.45	0.139100941442	0.139100363619	5.77823×10^{-7}
0.55	0.139100941442	0.139100363619	5.77823×10^{-7}
0.65	0.127619498329	0.127618902166	5.96163×10^{-7}
0.75	0.104787310536	0.104786680206	6.30330×10^{-7}
0.85	0.070859894772	0.070859219050	6.75722×10^{-7}
0.95	0.026206378688	0.026205652306	7.26382×10^{-7}

Table 2: $\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 10$.

x	*Exact solution*	*Numerical solution*	*Error*
0.05	0.059859139553	0.059856961344	2.17821×10^{-6}
0.15	0.163357650260	0.163356185591	1.46467×10^{-6}
0.25	0.243336567795	0.243335840391	7.27404×10^{-7}
0.35	0.297849968155	0.297849854031	1.14124×10^{-7}
0.45	0.325480651313	0.325480886663	2.35350×10^{-7}
0.55	0.325480651313	0.325480886663	2.35350×10^{-7}
0.65	0.297849968155	0.297849854031	1.14124×10^{-7}
0.75	0.243336567795	0.243335840391	7.27404×10^{-7}
0.85	0.163357650260	0.163356185591	1.46467×10^{-6}
0.95	0.059859139553	0.059856961344	2.17821×10^{-6}

5. Conclusion

In this paper non-polynomial cubic spline method is applied for solving the Bratu equation. The non-polynomial spline method reduce the computation of the Bratu equation to some nonlinear algebraic equations. The analytical results are illustrated with two numerical examples.

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