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Π_1-SURFACES OF BIHARMONIC CONSTANT Π_1-SLOPE CURVES ACCORDING TO TYPE-2 BISHOP FRAME IN THE SOL SPACE $\mathfrak{SOL}^3$

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Abstract. In this paper, we study $\mathbf{\Pi}_1-$ surfaces of biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop in the $\mathfrak{SOL}^3$. We characterize the $\mathbf{\Pi}_1-$ surfaces of biharmonic constant $\mathbf{\Pi}_1-$ slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the $\mathfrak{SOL}^3$.

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1. Introduction

A ruled surface can be generated by the motion of a line in space, similar to the way a curve can be generated by the motion of a point. A 3D surface is called ruled if through each of its points passes at least one line that lies entirely on that surface.

In this paper, we study $\mathbf{\Pi}_1-$ surfaces of biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop in the $\mathfrak{SOL}^3$. We characterize the $\mathbf{\Pi}_1-$ surfaces of biharmonic constant $\mathbf{\Pi}_1-$ slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the $\mathfrak{SOL}^3$.

2. Riemannian Structure of Sol Space $\mathfrak{SOL}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{SOL}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ [16].

Note that the Sol metric can also be written as:

$$g_{\mathfrak{SOL}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = e^z dx, \omega^2 = e^{-z} dy, \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, mathbfe_2 = e^z \frac{\partial}{\partial y}, \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

3. Biharmonic Constant $\mathbf{\Pi}_1$–Slope Curves according to New Type-2 Bishop Frame in Sol Space $\mathfrak{SOL}^3$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{T}) &= 1, g_{\mathfrak{SOL}^3}(\mathbf{N},\mathbf{N}) = 1, g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{B}) = 1, \\ g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{B}) = g_{\mathfrak{SOL}^3}(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{T}) &= 1, g_{\mathfrak{SOL}^3}(\mathbf{M}_1,\mathbf{M}_1) = 1, g_{\mathfrak{SOL}^3}(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{M}_1) &= g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{M}_2) = g_{\mathfrak{SOL}^3}(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\mathfrak{U}(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \mathfrak{U}'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Let γ be a unit speed regular curve in $\mathfrak{SOL}^3$ and (3.1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{\Pi}_1 &= -\epsilon_1\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{\Pi}_2 &= -\epsilon_2\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \epsilon_1\mathbf{\Pi}_1 + \epsilon_2\mathbf{\Pi}_2, \end{aligned} \tag{4}$$

where

$$\begin{aligned} g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{B}) &= 1, g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_1) = 1, g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_2,\mathbf{\Pi}_2) = 1, \\ g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_1) &= g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_2) = g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_2) = 0. \end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet–Serret frame, first we write

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}. \tag{3.5}$$

The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\begin{aligned} &\mathbf{T} = \sin\mathfrak{A}(s)\,\mathbf{\Pi}_1 - \cos\mathfrak{A}(s)\,\mathbf{\Pi}_2, \\ &\mathbf{N} = \cos\mathfrak{A}(s)\,\mathbf{\Pi}_1 + \sin\mathfrak{A}(s)\,\mathbf{\Pi}_2, \\ &\mathbf{B} = \mathbf{B}. \end{aligned}$$

So by (3.5), we may express

$$\begin{aligned} \epsilon_1 &= -\tau\cos\mathfrak{A}(s), \\ \epsilon_2 &= -\tau\sin\mathfrak{A}(s). \end{aligned}$$

By this way, we conclude

$$\mathfrak{A}(s) = \arctan\frac{\epsilon_2}{\epsilon_1}.$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathfrak{A}(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve γ. We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \epsilon_1, \epsilon_2\}$ as type-2 Bishop invariants of the curve γ, [22].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{\Pi}_1 &= \pi_1^1\mathbf{e}_1 + \pi_1^2\mathbf{e}_2 + \pi_1^3\mathbf{e}_3, \\ \mathbf{\Pi}_2 &= \pi_2^1\mathbf{e}_1 + \pi_2^2\mathbf{e}_2 + \pi_2^3\mathbf{e}_3. \\ \mathbf{B} &= B^1e_1 + B^2e_2 + B^3e_3, \end{aligned} \tag{5}$$

Theorem 1. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1$−slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the parametric equations of γ are*

$$\begin{aligned} \boldsymbol{x}(s) &= \int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2] \\ &\quad -\cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]ds, \qquad (6) \\ \boldsymbol{y}(s) &= \int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2] \\ &\quad -\cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]ds, \\ \boldsymbol{z}(s) &= -\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3, \end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are constants of integration, [13].

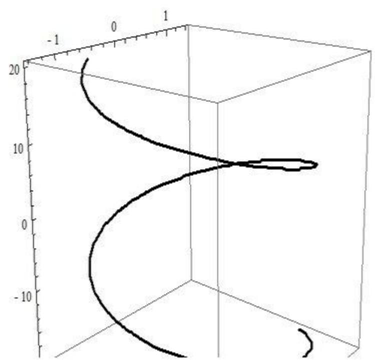

Fig. 1.

4. $\mathbf{\Pi}_1$− Surfaces of Biharmonic Constant $\mathbf{\Pi}_1$−Slope Curves according to New Type-2 Bishop Frame in Sol Space $\mathfrak{SOL}^3$

The purpose of this section is to study $\mathbf{\Pi}_1$− surfaces of biharmonic constant $\mathbf{\Pi}_1$−slope curves according to new type-2 Bishop frame in Sol space $\mathfrak{SOL}^3$

The $\mathbf{\Pi}_1$− surface of γ is a ruled surface

$$\mathcal{B}(s,u) = \gamma(s) + u\mathbf{\Pi}_1. \qquad (4.1)$$

Theorem 2. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1$−slope curve according to type-2 Bishop frame and $\mathcal{B}$ its $\mathbf{\Pi}_1$− surface in the*

$\mathfrak{SOL}^3$. Then, the equation of $\mathcal{B}$ is

$$\begin{aligned}\mathcal{B}(s,u) &= [e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3}\int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3} \\ &\quad [\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]-\cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]ds \\ &\quad +u\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]\mathbf{e}_1 \\ &\quad +[e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3}\int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3} \\ &\quad [\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]-\cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]ds \\ &\quad +u\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]\mathbf{e}_2 \\ &\quad +[-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3+u\cos\mathfrak{E}]\mathbf{e}_3,\end{aligned} \tag{7}$$

where $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ are constants of integration.

Proof. We assume that γ is a unit speed biharmonic constant $\mathbf{\Pi}_1-$slope curve according to type-2 Bishop frame and in the $\mathfrak{SOL}^3$.

The vector $\mathbf{\Pi}_1$ is a unit vector, we have the following equation

$$\mathbf{\Pi}_1=\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]\,\mathbf{e}_1+\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]\,\mathbf{e}_2+\cos\mathfrak{E}\mathbf{e}_3, \tag{4.3}$$

where $\mathcal{R}_1,\mathcal{R}_2\in\mathbb{R}$.

Substituting (4.3) to (4.1), we have (4.2). Thus, the proof is completed.

We can prove the following interesting main result.

Theorem 3. *Let $\gamma: I\longrightarrow\mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1-$slope curve according to type-2 Bishop frame and $\mathcal{B}$ its $\mathbf{\Pi}_1-$ surface in the $\mathfrak{SOL}^3$. Then, the parametric equations of $\mathcal{B}$ are*

$$\begin{aligned}\boldsymbol{x}_{\mathcal{B}}(s,u) &= e^{-[-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3+u\cos\mathfrak{E}]}[e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3} \\ &\quad \int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2] \\ &\quad -\cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]ds+u\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]], \\ \boldsymbol{y}_{\mathcal{B}}(s,u) &= e^{[-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3+u\cos\mathfrak{E}]}[e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3} \\ &\quad \int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2] \\ &\quad -\cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]ds+u\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]], \\ \boldsymbol{z}_{\mathcal{B}}(s,u) &= -\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3+u\cos\mathfrak{E},\end{aligned} \tag{8}$$

where $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ are constants of integration.

Proof. The parametric equations of $\mathcal{B}$ can be found from (4.1), (4.2). This concludes the proof of Theorem.

From above theorem, we have

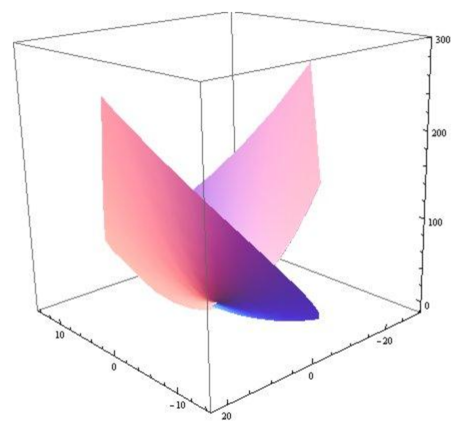

Fig. 2.

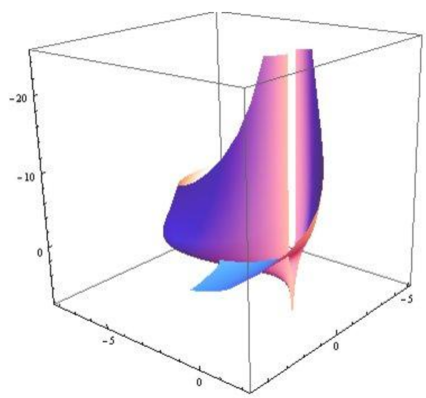

Fig. 3.

Thus, we proved the following:

Theorem 4. *Let* $\gamma : I \longrightarrow \mathfrak{SOL}^3$ *be a unit speed non-geodesic biharmonic constant* $\mathbf{\Pi}_1-$*slope curve according to type-2 Bishop frame and* $\mathcal{B}$ *its* $\mathbf{\Pi}_1-$ *surface in the* $\mathfrak{SOL}^3$. *Then, normal of* $\mathcal{B}$ *is*

$$\begin{aligned}\mathbf{N}_{\mathcal{B}} &= [-\cos[\kappa s]\sin[\mathcal{R}_1 s+\mathcal{R}_2] - u\epsilon_1\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]\mathbf{e}_1 \\ &\quad +[\cos[\kappa s]\cos[\mathcal{C}_1 s+\mathcal{C}_2] - u\epsilon_1\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]\mathbf{e}_2 + u\epsilon_1\sin\mathfrak{E}\mathbf{e}_3,\end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2$ *are constants of integration.*

Corollary 4.3. *Let* $\gamma : I \longrightarrow \mathfrak{SOL}^3$ *be a unit speed non-geodesic biharmonic constant* $\mathbf{\Pi}_1-$*slope curve according to type-2 Bishop frame and* $\mathcal{B}$ *its* $\mathbf{\Pi}_1-$ *surface in the* $\mathfrak{SOL}^3$. *Then, normal of* $\mathcal{B}$ *is*

$$\begin{aligned} \mathbf{N}_{\mathcal{B}} &= [-\cos[\kappa s]\sin[\mathcal{R}_1 s + \mathcal{R}_2] - u\epsilon_1 \cos\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_1 \\ &\quad + [\cos[\kappa s]\cos[\mathcal{C}_1 s + \mathcal{C}_2] - u\epsilon_1 \cos\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_2 + u\epsilon_1 \sin\mathfrak{E}\mathbf{e}_3, \end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2$ *are constants of integration.*

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