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THE SOLUTIONS OF HOMOGENEOUS AND NONHOMOGENEOUS LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS BY VARIATIONAL ITERATION METHOD

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Abstract. In this study, we investigate solutions of homogeneous and nonhomogeneous linear Fractional Differential Equations (FDE) by means of Variational Iteration Method (VIM). We show that homogenous and nonhomogeneous linear FDE be solved by means of VIM. Then, we draw 2D graphics of these equations by means of programming language Mathematica. The results of the considered equations which solve by VIM reveal that this method is very effective and convenient for solving such equations of homogeneous and nonhomogeneous linear FDE. A useful side of this method is that it easily implement to variety problems such as homogeneous, nonhomogeneous, linear and nonlinear fractional differential equations.

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1. Introduction

Fractional differential equations (FDE) which are homogeneous or nonhomogeneous [1] do not have analytical solutions, therefore samy-analytical methods have to be used to obtain approximate solution. One of the most important these technics is Variational Iteration Method (VIM) [2,3] submitted by J.H.He [4,5]. He have used to obtain approximate solution a lot of differential equations which are homogeneous and nonhomogeneous. The techniques have a lot of advantages over the classical techniques [6] because the method does not need linearization. It is providing an sufficient solution with high accuracy, minimal calculation. In recent years, a few paper has been submitted about fractional differential equations which have been solved by VIM [7-10]. In this paper, we have used to VIM for solving homogeneous and nonhomogeneous fractional differential equations [1].

2. Variational Iteration Method (VIM)

In order to illustrate the basic concepts of VIM, the following nonlinear partial differential equation can be considered

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \tag{1}$$

where L is a linear time derivative operator, R is a linear operator which has derivatives with respect to x and t , N is a nonlinear differential operator, and f(x,t) is a source nonhomogeneous term. According to the VIM, we can construct the following iteration formula for Eq.(1)

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{Lu_n(x,\tau) + R\widetilde{u_n}(x,\tau) + N\widetilde{u_n}(x,\tau) - f(x,\tau)\}d\tau, \tag{2}$$

where the parameter τ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, the subscript n denotes the n th-order approximation, and $\widetilde{u_n}$ is considered as a restricted variation which means $\delta\widetilde{u_n} = 0$.

It is obvious now that the main steps of He's VIM require first the determination of the Lagrange's multiplier λ that will be identified optimally. Having determined λ the successive approximations u_{n+1} $(n > 0)$ of the solution u will be obtained upon using suitable selective function which satisfies the boundary conditions. It can generally identify that Lagrange's multiplier is as following:

$$\lambda(s,t) = (-1)^m \frac{(s-t)^{(m-1)\alpha}}{\Gamma[1+(m-1)\alpha]}$$

where is $y^{(m\alpha)} = {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha \cdots {}_aD_x^\alpha y = {}_aD_x^{m\alpha} y$, $m \in Z$, $0 < \alpha < 1$.

3. Applications of VIM

Example 1

Firstly, we consider the homogenous FDE [1] as following:

$$D_t^\alpha y(t) = -y(t), \tag{3}$$

with the condition

$$y(0) = 1, t > 0, 0 < \alpha \leq 1. \tag{4}$$

According to variational iteration method, we construct a structure of iteration for Eq.(3) in the following form

$$y_{k+1}(t) = y_k(t) + \int_0^t \lambda(\tau)[\frac{\partial^\alpha y_k(\tau)}{\partial \tau^\alpha} + y_k(\tau)]d\tau, k = 0, 1, 2, \cdots . \tag{5}$$

Calculating variation with respect to y_k , we have the stationary conditions as following;

$$1 + \lambda(\tau)|_{\tau=t} = 0, \quad \lambda'(\tau) = 0 \tag{6}$$

The Lagrange multiplier can be easily identified as;

$$\lambda(\tau) = -1, \tag{7}$$

Substituting Eq.(7) into Eq.(5), we have the iteration formula as following

$$y_{k+1}(t) = y_k(t) - \int_0^t [\frac{\partial^\alpha y_k(\tau)}{\partial \tau^\alpha} + y_k(\tau)]d\tau. \tag{8}$$

When we choose for the initial approximate solution in the form of $y(0) = 1$ and substituting into Eq.(8) and constructing a simple algorithm in any computational software such as Mathematica, we get the approximate solution as following;

$$for \quad k = 0, \quad y_1 = y_0(t) - \int_0^t [\frac{\partial^\alpha y_0(\tau)}{\partial \tau^\alpha} + y_0(\tau)]d\tau = 1 - \int_0^t [\frac{\partial^\alpha 1}{\partial \tau^\alpha} + 1]d\tau,$$

$$= 1 - t - \int_0^t [\frac{\partial^\alpha 1}{\partial \tau^\alpha}]d\tau = 1 - t - (\frac{t^{1-\alpha}}{\Gamma[2-\alpha]} - 1),$$

$$= 2 - t - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]}. \tag{9}$$

$$for \quad k = 1, \quad y_2 = y_1(t) - \int_0^t [\frac{\partial^\alpha y_1(\tau)}{\partial \tau^\alpha} + y_1(\tau)]d\tau,$$

$$= 2 - t - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]} - \int_0^t [\frac{\partial^\alpha (2 - t - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]})}{\partial \tau^\alpha} + y_1]d\tau,$$

$$= 2 - 3t + \frac{t^2}{2} - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]} + \frac{t^{2-\alpha}}{\Gamma[3-\alpha]} - I_1,$$

where

$$I_1 = \int_0^t [\frac{\partial^\alpha y_1}{\partial \tau^\alpha}] d\tau = \int_0^t [\frac{\partial^\alpha (2 - t - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]})}{\partial \tau^\alpha}] d\tau,$$

$$= \int_0^t [\frac{\partial^\alpha 2}{\partial \tau^\alpha}] d\tau - \int_0^t [\frac{\partial^\alpha \tau}{\partial \tau^\alpha}] d\tau - \frac{1}{\Gamma[2-\alpha]} \int_0^t [\frac{\partial^\alpha \tau^{1-\alpha}}{\partial \tau^\alpha}] d\tau,$$

$$= -2 - t + \frac{3t^{1-\alpha}}{\Gamma[2-\alpha]} - \frac{\Gamma[2-\alpha]}{\Gamma[3-2\alpha]}(t^{2-2\alpha} - t^{1-\alpha}),$$

thus

$$y_2(t) = 4 - 2t + \frac{t^2}{2} - \frac{4t^{1-\alpha}}{\Gamma[2-\alpha]} + \frac{t^{2-\alpha}}{\Gamma[3-\alpha]} + \frac{\Gamma[2-\alpha]}{\Gamma[3-2\alpha]}(t^{2-2\alpha} - t^{1-\alpha}). \quad (10)$$

The rest of components of the iteration formula Eq.(8) can be obtained in the same manner using programming of Mathematica. The exact solution of Eq.(3) is given by;

$$y(t) = E_\alpha(-t^\alpha) = \Sigma_{k=0}^{\infty} \frac{(-t^\alpha)^k}{\Gamma[\alpha k + 1]}. \quad (11)$$

When we take k=2 at two steps for VIM, we obtain that exact solution of Eq(3) as following:

$$y(t) = 1 - \frac{t^\alpha}{\Gamma[1+\alpha]} + \frac{t^{2\alpha}}{\Gamma[1+2\alpha]}. \quad (12)$$

Example 2

Secondly, we consider the nonhomogeneous FDE [1] defined by following equation:

$$D_t^\alpha y(t) = \frac{2t^{2-\alpha}}{\Gamma[3-\alpha]} - \frac{t^{1-\alpha}}{\Gamma[2-\alpha]} - y(t) + t^2 - t, \quad (13)$$

with the condition

$$y(0) = 0, \quad t > 0, \quad 0 < \alpha \leq 1. \quad (14)$$

According to variational iteration method, we construct a structure of Eq.(13) in the as following form:

$$y_{k+1} = y_k + \int_0^t \lambda(\tau) [\frac{\partial^\alpha y_k(\tau)}{\partial \tau^\alpha} - \frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]} + \frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]} + y_k(\tau) - \tau^2 + \tau] d\tau. \quad (15)$$

Calculating variation with respect to y_n , we have the stationary conditions as following;

$$1+\lambda(\tau)|_{\tau=t}=0, \qquad \lambda'(\tau)=0 \tag{16}$$

The Lagrange multiplier can be easily identified as:

$$\lambda(\tau)=-1, \tag{17}$$

Substituting Eq.(17) into Eq.(15), we have the iteration formula as following:

$$y_{k+1}=y_k(t)-\int_0^t[\frac{\partial^\alpha y_k(\tau)}{\partial\tau^\alpha}-\frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]}+y_k(\tau)-\tau^2+\tau]d\tau. \tag{18}$$

When we choose for the initial approximate solution in the form of $y(0)=0$ and substituting into Eq.(18) and constructing a simple algorithm in any computational software such as Mathematica, we get the approximate solution as following:

$$\begin{aligned} for \quad k=0,\ y_1 &= y_0(t)-\int_0^t[\frac{\partial^\alpha y_0(\tau)}{\partial\tau^\alpha}-\frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]}+y_0(\tau)-\tau^2+\tau]d\tau,\\ &= \int_0^t[\frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]}-\frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]}+\tau^2-\tau]d\tau,\\ &= \frac{2t^{3-\alpha}}{\Gamma[4-\alpha]}-\frac{t^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{t^3}{3}-\frac{t^2}{2}. \end{aligned} \tag{19}$$

$$\begin{aligned} for \quad k=1,\ y_2 &= y_1(t)-\int_0^t[\frac{\partial^\alpha y_1(\tau)}{\partial\tau^\alpha}-\frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]}+y_1(\tau)-\tau^2+\tau]d\tau,\\ &= \frac{2t^{3-\alpha}}{\Gamma[4-\alpha]}-\frac{t^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{t^3}{3}-\frac{t^2}{2}-\int_0^t[\frac{\partial^\alpha y_1(\tau)}{\partial\tau^\alpha}]d\tau\\ &\quad -\int_0^t[-\frac{2\tau^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{\tau^{1-\alpha}}{\Gamma[2-\alpha]}+y_1(\tau)-\tau^2+\tau]d\tau,\\ y_2(t) &= \frac{-3t^2}{2}+\frac{7t^3}{6}-\frac{t^4}{12}-\frac{3t^{2-\alpha}}{\Gamma[3-\alpha]}+\frac{8t^{3-\alpha}}{\Gamma[4-\alpha]}\\ &\quad -\frac{4t^{4-\alpha}}{\Gamma[5-\alpha]}+\frac{t^{3-2\alpha}}{\Gamma[4-2\alpha]}-\frac{2t^{4-2\alpha}}{\Gamma[5-2\alpha]}. \end{aligned} \tag{20}$$

The rest of components of the iteration formula Eq.(18) can be obtained in the same manner using programming of Mathematica. The exact solution of Eq.(13) is given by

$$y(t)=t^2-t. \tag{21}$$

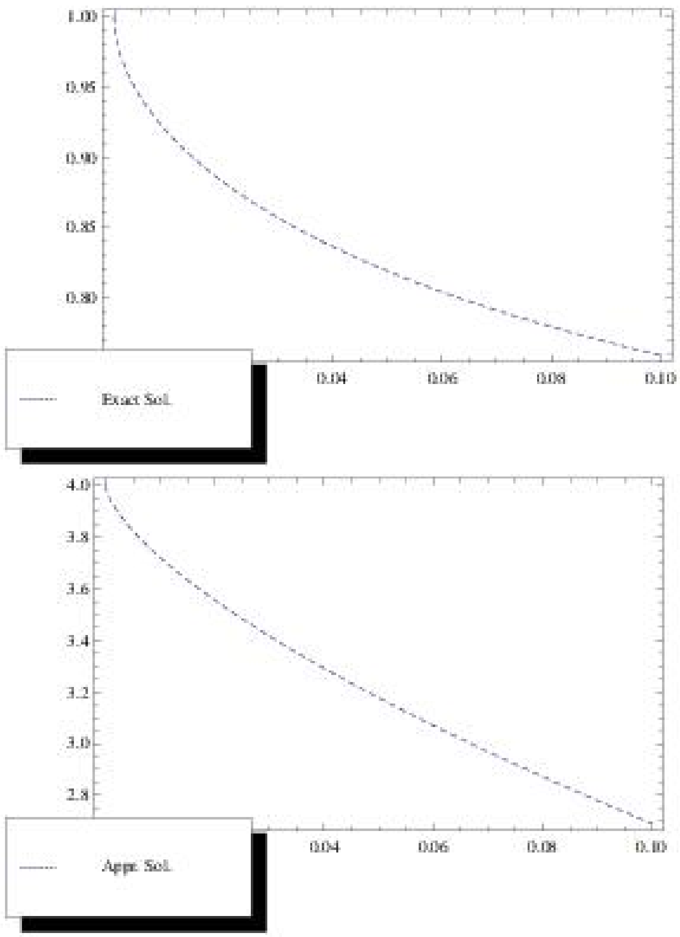

Figure 1: 2D surfaces of the solution (12) which is exact approximate solution and Eq.(10) being approximate solution obtained by using VIM have been shown at $\alpha = 0.35$, and $0 < t < 0.1$.

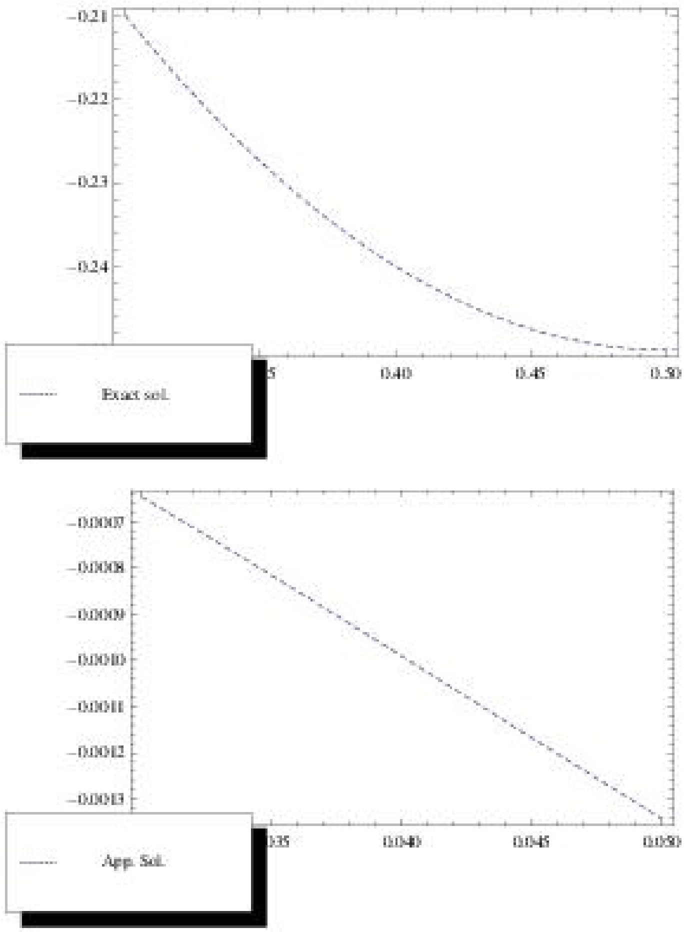

Figure 2: 2D surfaces of the solution (21) which is exact approximate solution and Eq.(20) being approximate solution obtained by using VIM have been shown at $\alpha = 0.1$, and $0.03 < t < 0.05$.

4. Conclusions

In this paper, the variational iteration method has successfully been applied for solving two equations of homogenous and nonhomogeneous linear FDE. Approximate solutions of these equations are very suitable in terms of provides the equation. Consequently, the results of equations tell us that the this method can be alternative ways for solving of such equations of homogenous and nonhomogenous FDEs.

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