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ON CERTAIN SUBCLASSES OF HOLOMORPHIC FUNCTIONS DEFINED ON THE UNIT DISK

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Abstract. We present several results for certain subclasses of the uniformly α spirallike functions. These include distortion and covering theorems, extreme points, radii of close-to-convexity, starlikeness and convexity for these classes. We also obtain integral means inequalities with the extremal functions for these classes.

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1. Introduction, definition and preliminaries

Let A denote the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are regular in the unit disk $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. The function $f \in A$ is spirallike if $Re \left\{ e^{-i\alpha} \frac{zf'(z)}{f(z)} \right\} > 0$ for all $z \in \Delta$ and for some α with $|\alpha| < \pi/2$. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike.

The class of uniformly convex functions was introduced and studied by various authors as in [1, 2, 4, 5, 6].

Let T denote the class consisting of functions f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where a_n is a non-negative real number.

Silverman [9] introduced and investigated many subclasses of T.

We now defined $UCSPT(\alpha, \beta)$ and $SP_P T(\alpha, \beta)$.

Definition 1. *[7] Let* $UCSPT(\alpha,\beta)$ *be the class of functions* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *which satisfy the condition*

$$Re\ e^{-i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right| + \beta,$$

$|\alpha| < \pi/2,\ 0 \leq \beta < 1.$

Definition 2. *[7] Let* $SP_PT(\alpha,\beta)$ *be the class of functions* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *which satisfy the condition*

$$Re\ e^{-i\alpha}\frac{zf'(z)}{f(z)} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta,$$

$|\alpha| < \pi/2,\ 0 \leq \beta < 1.$

In this paper we discuss several results for the classes $UCSPT(\alpha,\beta)$ and $SP_PT(\alpha,\beta)$ like distortion bounds, extreme points, radii of close-to-convexity, starlikeness and convexity. We also obtain integral means inequality for the functions belonging to this class.

For proving our results we require the following lemmas.

Lemma 1. *[7] Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. *Then*

$$\sum_{n=2}^{\infty}(2n - cos\ \alpha - \beta)na_n \leq cos\ \alpha - \beta.$$

if and only if $f(z)$ *is in* $UCSPT(\alpha,\beta)$.

Lemma 2. *[7]* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ *is in* $SP_PT(\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}(2n - cos\ \alpha - \beta)a_n \leq cos\ \alpha - \beta.$$

2. Distortion and covering theorems

Theorem 3. *If* $f(z) \in UCSPT(\alpha,\beta)$ *then*

$$r - \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)}\ r^2 \leq |f(z)| \leq r + \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)}\ r^2$$

and

$$1-\frac{cos\ \alpha-\beta}{4-cos\ \alpha-\beta}\ r\le|f'(z)|\le 1+\frac{cos\ \alpha-\beta}{4-cos\ \alpha-\beta}\ r$$

and the extreme points are

$$f_1(z)=z,\quad f_n(z)=z-\frac{cos\ \alpha-\beta}{n(2n-cos\ \alpha-\beta)}\ z^n,\quad n=2,3,\ldots$$

The result is sharp for $f(z)=z-\dfrac{cos\ \alpha-\beta}{2(4-cos\ \alpha-\beta)}\ z^2,\ z=\pm r.$

Proof. $f(z)\in UCSPT(\alpha,\beta)$. Hence by Lemma 1

$$\sum_{n=2}^{\infty}(2n-cos\ \alpha-\beta)na_n\le cos\ \alpha-\beta.$$

$$\therefore\ \sum_{n=2}^{\infty}a_n\le\frac{cos\ \alpha-\beta}{2(4-cos\ \alpha-\beta)}$$

From $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ with $|z|=r$ $(r<1)$ we have

$$\begin{aligned}|f(z)|&\le r+\sum_{n=2}^{\infty}a_nr^n\\&\le r+\sum_{n=2}^{\infty}a_nr^2\\&\le r+\frac{cos\ \alpha-\beta}{2(4-cos\ \alpha-\beta)}r^2.\end{aligned}$$

Theorem 4. *If* $f(z)\in SP_PT(\alpha,\beta)$ *then*

$$r-\frac{cos\ \alpha-\beta}{4-cos\ \alpha-\beta}r^2\le|f(z)|\le r+\frac{cos\ \alpha-\beta}{4-cos\ \alpha-\beta}r^2.$$

The result is sharp for $f(z)=z-\dfrac{cos\ \alpha-\beta}{4-cos\ \alpha-\beta}\ z^2,\ z=\pm r.$

Proof. From Lemma 2,

$$\sum_{n=2}^{\infty}(2n-cos\ \alpha-\beta)a_n\le cos\ \alpha-\beta.$$

$$\therefore \quad \sum_{n=2}^{\infty} a_n \leq \frac{cos\ \alpha - \beta}{4 - cos\ \alpha - \beta}$$

From $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ with $|z| = r$ $(r < 1)$ we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \\ &\leq r + \sum_{n=2}^{\infty} a_n r^2 \\ &\leq r + \frac{cos\ \alpha - \beta}{4 - cos\ \alpha - \beta} r^2. \end{aligned}$$

Also

$$1 - \frac{2(cos\ \alpha - \beta)}{4 - cos\ \alpha - \beta}\ r \leq |f'(z)| \leq 1 + \frac{2(cos\ \alpha - \beta)}{4 - cos\ \alpha - \beta}\ r$$

and the extreme points are

$$f_1(z) = z, \quad f_n(z) = z - \frac{cos\ \alpha - \beta}{2n - cos\ \alpha - \beta}\ z^n, \quad n = 2, 3, \ldots$$

3. Integral means inequalities

In [9], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjectured in [10] and settled in [11], that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta, \text{ for all } f \in T, \eta > 0 \text{ and } 0 < r < 1.$$

In [11], he also proved his conjecture for some subclasses of T.

Now, we prove Silverman's conjecture for the class of functions $UCSPT(\alpha, \beta)$. An analogous result is also obtained for the family of functions $SP_PT(\alpha, \beta)$.

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3].

Two given functions f and g, which are analytic in Δ, the function f is said to be subordinate to g in Δ if there exists a function w analytic in Δ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z))$ $(z \in \Delta)$. We denote this subordination by $f(z) \prec g(z)$.

Lemma 5. *If the functions f and g are analytic in D with $f(z) \prec g(z)$ then for $\eta > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta.$$

Now we discuss the integral means inequalities for $UCSPT(\alpha, \beta)$.

Theorem 6. *Let $f \in UCSPT(\alpha, \beta)$, $|\alpha| < \pi/2$, $0 \leq \beta < 1$ and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta \tag{2}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (2) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)} z\right|^\eta d\theta$$

By Lemma 2 it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{cos\ \alpha - \beta}{2(4 - cos\ \alpha - \beta)} w(z)$$

and using $\sum_{n=2}^{\infty} (2n - cos\ \alpha - \beta) n a_n \leq cos\ \alpha - \beta$ we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{2(4 - cos\ \alpha - \beta)}{cos\ \alpha - \beta} a_n z^{n-1}\right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{n(2n - cos\ \alpha - \beta)}{cos\ \alpha - \beta} a_n \leq |z|. \end{aligned}$$

This completes the proof by Lemma 1.

For completeness, we now give the integral means inequality for the class $SP_PT(\alpha,\beta)$. The method of proving Theorem 7 is similar to that of Theorem 6.

Theorem 7. *Let* $f \in SP_PT(\alpha,\beta)$, $|\alpha| < \pi/2$, $0 \le \beta < 1$ *and* $f_2(z)$ *is defined by* $f(z) = z - \dfrac{cos\ \alpha - \beta}{4 - cos\ \alpha - \beta} z^2$. *Then for* $z = re^{i\theta}$, $0 < r < 1$ *we have*

$$\int_0^{2\pi} |f(z)|^\eta d\theta \le \int_0^{2\pi} |f_2(z)|^\eta d\theta.$$

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 8. *If* $f(z) \in UCSPT(\alpha,\beta)$ *then* f *is close-to-convex of order* γ $(0 \le \gamma < 1)$ *in* $|z| < r_1(\alpha,\beta,\gamma)$ *where*

$$r_1(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{(1-\gamma)(2n - cos\ \alpha - \beta}{cos\ \alpha - \beta} \right\}^{\frac{1}{n-1}}, \quad n \ge 2.$$

Proof. By a computation we have

$$|f'(z) - 1| = \left| - \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Now, f is close-to-convex of order γ if we have the condition

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\gamma} \right) a_n |z|^{n-1} \le 1. \tag{3}$$

Considering the coefficient conditions required by Lemma 1 the above inequality (3) is true if $\left(\dfrac{n}{1-\gamma} \right) |z|^{n-1} \le \dfrac{n(2n - cos\ \alpha - \beta)}{cos\ \alpha - \beta}$ or if

$$|z| \le \left\{ \frac{(1-\gamma)(2n - cos\ \alpha - \beta)}{cos\ \alpha - \beta} \right\}^{\frac{1}{n-1}}, \quad n \ge 2.$$

This expression yields the bounds required by the above theorem.

Theorem 9. *If* $f(z) \in UCSPT(\alpha,\beta)$ *then* f *is starlike of order* γ $(0 \le \gamma < 1)$ *in* $|z| < r_2(\alpha,\beta,\gamma)$ *where*

$$r_2(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{(1-\gamma)n(2n - cos\ \alpha - \beta)}{(n-\gamma)(cos\ \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \ge 2.$$

Proof. By a computation we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|-\frac{\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Now f is starlike of order γ if we have the condition

$$\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1-\gamma}\right)a_n|z|^{n-1} \leq 1. \tag{4}$$

Considering the coefficient conditions required by Lemma 1, the above inequality is true if $\left(\frac{n-\gamma}{1-\gamma}\right)|z|^{n-1} \leq \frac{n(2n-cos\ \alpha-\beta)}{cos\ \alpha-\beta}$ or if

$$|z| \leq \left\{\frac{(1-\gamma)n(2n-cos\ \alpha-\beta)}{(n-\gamma)(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

This last expression yields the bound required.

Theorem 10. *If $f(z) \in UCSPT(\alpha,\beta)$ then f is convex of order γ $(0 \leq \gamma < 1)$ in $|z| < r_3(\alpha,\beta,\gamma)$ where*

$$r_3(\alpha,\beta,\gamma) = \inf_n \left\{\frac{(1-\gamma)(2n-cos\ \alpha-\beta)}{(n-\gamma)(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof. By a computation we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|-\frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n|z|^{n-1}}.$$

Now f is convex of order γ if we have the condition

$$\sum_{n=2}^{\infty}\frac{n(n-\gamma)}{1-\gamma}a_n|z|^{n-1} \leq 1. \tag{5}$$

Considering the coefficient conditions required by Lemma 1, the above inequality (5) is true if $\left(\frac{n(n-\gamma)}{1-\gamma}\right)|z|^{n-1} \leq \frac{n(2n-cos\ \alpha-\beta)}{cos\ \alpha-\beta}$ or if

$$|z| \leq \left\{\frac{(1-\gamma)(2n-cos\ \alpha-\beta)}{(n-\gamma)(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

This gives the bound required by the above theorem.

For completeness, we give without proof, theorems concerning the radii of close-to-convexity, starlikeness and convexity for the class $SP_PT(\alpha,\beta)$.

Theorem 11. *If $f(z) \in SP_PT(\alpha,\beta)$ then f is close-to-convex of order γ $(0 \leq \gamma < 1)$ in $|z| < r_4(\alpha,\beta,\gamma)$ where*

$$r_4(\alpha,\beta,\gamma) = \inf_n \left\{\frac{(1-\gamma)(2n-cos\ \alpha-\beta)}{n(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Theorem 12. *If $f(z) \in SP_PT(\alpha,\beta)$ then f is starlike of order γ $(0 \leq \gamma < 1)$ in $|z| < r_5(\alpha,\beta,\gamma)$ where*

$$r_5(\alpha,\beta,\gamma) = \inf_n \left\{\frac{(1-\gamma)(2n-cos\ \alpha-\beta)}{(n-\gamma)(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Theorem 13. *If $f(z) \in SP_PT(\alpha,\beta)$ then f is convex of order γ $(0 \leq \gamma < 1)$ in $|z| < r_6(\alpha,\beta,\gamma)$ where*

$$r_6(\alpha,\beta,\gamma) = \inf_n \left\{\frac{(1-\gamma)(2n-cos\ \alpha-\beta)}{n(n-\gamma)(cos\ \alpha-\beta)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

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