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HILBERT-SCHMIDT SEQUENCES AND DUAL OF G-FRAMES

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Abstract. In this paper, we characterize the dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$ and show that each dual is precisely the sequence $\{\Theta_i\}_{i=1}^{\infty} = \{\phi_i^* V^*\}_{i=1}^{\infty}$, where the operator $V : l^2(\mathbb{N} \times \mathbb{N}) \to H$, is a bounded left inverse of the analysis operator of the frame induced by $\{\Lambda_i\}_{i=1}^{\infty}$ and for each $i \in \mathbb{N}$, ϕ_i is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N} \times \mathbb{N})$. Also, we prove that every Hilbert-Schmidt sequence is a g-Bessel sequence and the composition of synthesis operator with analysis operator of a Hilbert-Schmidt sequence is a trace class operator.

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1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer in [5] to study some deep problems in nonharmonic Fourier series. After the fundamental paper by Daubechies, Grossman and Meyer [4], frame theory began to be widely used, particularly in the more specialized context of wavelet frames [7]. The concept of g-frames which was first presented by Sun in [13], includes many other generalizations of frames, e.g., outer frames [1] and oblique frames [3, 6]. For more details, we refer to [8, 11, 13].

Throughout this paper, H and K are complex separable Hilbert spaces and $\{H_i\}_{i\in I}$ is a sequence of closed subspaces of K. I and L are subsets of $\mathbb{Z}$, and for each $i \in I$, J_i is a subset of $\mathbb{Z}$. $L(H, H_i)$ is the collection of all bounded linear operators of H into H_i.

This paper is organized as follows: In section 2, we recall some definitions and properties about g-frames which will be used in this paper. In section 3, we characterize the dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$ and show that every dual is precisely the sequence $\{\Theta_i\}_{i=1}^{\infty} = \{\phi_i^* V^*\}_{i=1}^{\infty}$, where $V : l^2(\mathbb{N} \times \mathbb{N}) \to H$, is a bounded left inverse of the analysis operator of the frame induced by $\{\Lambda_i\}_{i=1}^{\infty}$, and for each $i \in \mathbb{N}$, ϕ_i is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N}\times\mathbb{N})$. In section 4, we obtain some useful properties of g-Riesz bases and show that under some conditions every g-Riesz basis has a g-biorthogonal sequence and is a g-minimal frame. We define the concept of Hilbert-Schmidt sequences in section 5 and show that every Hilbert-Schmidt sequence is a g-Bessel sequence but the converse is not true, when H is an infinite dimensional Hilbert space. Also, we prove that the composition of synthesis operator with analysis operator of a Hilbert-Schmidt sequence is a trace class operator.

Definition 1. *[13] We call a sequence $\{\Lambda_i \in L(H, H_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H. \tag{1}$$

A and B are called the lower and upper g-frame bounds, respectively. The sequence $\{\Lambda_i\}_{i\in I}$ is called a g-Bessel sequence with bound B, if the second inequality in (1), satisfies.

We call $\{\Lambda_i\}_{i\in I}$ an exact g-frame if it ceases to be a g-frame whenever any of its elements is removed.

We say that $\{\Lambda_i\}_{i\in I}$ is g-complete, if $\overline{span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$.

We say that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$, if it satisfies the following assertions:

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* g_{i_2}\rangle = \delta_{i_1,i_2}\langle f_{i_1}, g_{i_2}\rangle, \quad i_1, i_2 \in I, \quad f_{i_1} \in H_{i_1}, g_{i_2} \in H_{i_2},$$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \ f \in H. \tag{2}$$

Remark 1. *We note that if $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis, then by (2), for each $f \in H$,*

$$\langle f, f\rangle = \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in I}\langle \Lambda_i^*\Lambda_i f, f\rangle = \langle \sum_{i\in I} \Lambda_i^*\Lambda_i f, f\rangle.$$

So, $f = \sum_{i\in I} \Lambda_i^*\Lambda_i f$.

For each sequence $\{H_i\}_{i\in I}$, we define the space

$$(\sum_{i\in I} \bigoplus H_i)_{l_2} = \{\{f_i\}_{i\in I} : f_i \in H_i, \ i \in I \ and \ \sum_{i\in I} \|f_i\|^2 < \infty\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle.$$

It is clear that $(\sum_{i\in I} \bigoplus H_i)_{l_2}$ is a Hilbert space.

Remark 2. *Suppose that for each $i \in I$, $\{e_{i,j}\}_{j\in J_i}$ is an orthonormal basis for H_i. For each $i \in I$ and $j \in J_i$, we define $E_{i,j} = \{\delta_{i,k} e_{i,j}\}_{k\in I}$, where $\delta_{i,k}$ is the Kronecker delta. Then $\{E_{i,j}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $(\sum_{i\in I} \bigoplus H_i)_{l_2}$ and for each $\{f_k\}_{k\in I} \in (\sum_{i\in I} \bigoplus H_i)_{l_2}$, we have*

$$\langle \{f_k\}_{k\in I}, E_{i,j}\rangle = \langle f_i, e_{i,j}\rangle.$$

We define the synthesis operator for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ as follows:

$$T_\Lambda : (\sum_{i\in I} \bigoplus H_i)_{l_2} \to H, \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i,$$

the series converges unconditionally in the norm of H. It is easy to show that the adjoint operator of T_Λ is as follows:

$$T_\Lambda^* : H \to (\sum_{i\in I} \bigoplus H_i)_{l_2}, \quad T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I},$$

T_Λ^* is called the analysis operator for $\{\Lambda_i\}_{i\in I}$. In [13], the g-frame operator S_Λ for a g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ is defined as follows:

$$S_\Lambda : H \to H, \quad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

Hence we have $S_\Lambda = T_\Lambda T_\Lambda^*$. If $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$ with bounds A and B, then the g-frame operator $S_\Lambda : H \to H$ is a bounded, self adjoint and invertible operator. The canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$ is defined by $\{\tilde{\Lambda}_i\}_{i\in I}$ where for each $i \in I$, $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ which is also a g-frame for H with respect to $\{H_i\}_{i\in I}$ with frame bounds B^{-1} and A^{-1}. Also we have

$$f = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in I} \tilde{\Lambda}_i^{\,*} \Lambda_i f, \quad f \in H.$$

Let for each $i \in I$, $\Lambda_i \in L(H, H_i)$. Suppose that for each $i \in I$, $\{e_{i,j}\}_{j\in J_i}$ is an orthonormal basis for H_i. Then

$$f \mapsto \langle \Lambda_i f, e_{i,j}\rangle,$$

defines a bounded linear functional on H. Consequently, for each $i \in I$ and $j \in J_i$, we can find $u_{i,j} \in H$ such that for each $f \in H$, $\langle f, u_{i,j}\rangle = \langle \Lambda_i f, e_{i,j}\rangle$. Hence

$$\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}, \quad f \in H,$$

and

$$\Lambda_i^* g = \sum_{j \in J_i} \langle g, e_{i,j} \rangle u_{i,j}, \quad i \in I, \quad g \in H_i.$$

In particular,

$$u_{i,j} = \Lambda_i^* e_{i,j}, \quad i \in I, \ j \in J_i. \tag{3}$$

We call $\{u_{i,j}\}_{i \in I, j \in J_i}$ the sequence induced by $\{\Lambda_i\}_{i \in I}$ with respect to $\{e_{i,j}\}_{i \in I, j \in J_i}$.

2. Characterization of dual of G-Frames

Definition 2. *[12] Let $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ be g-Bessel sequences for H with respect to $\{H_i\}_{i \in I}$. $\{\Theta_i\}_{i \in I}$ is called a dual g-frame of $\{\Lambda_i\}_{i \in I}$, if*

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i f, \quad f \in H.$$

The space $l^2(\mathbb{N} \times \mathbb{N})$ defined by

$$l^2(\mathbb{N} \times \mathbb{N}) = \{\{a_{i,j}\}_{i,j=1}^{\infty} : \ a_{i,j} \in \mathbb{C}, \ \sum_{i,j=1}^{\infty} |a_{i,j}|^2 < \infty\},$$

with inner product given by

$$\langle \{a_{i,j}\}_{i,j=1}^{\infty}, \{b_{i,j}\}_{i,j=1}^{\infty} \rangle = \sum_{i,j=1}^{\infty} \langle a_{i,j}, b_{i,j} \rangle,$$

is a Hilbert space. For each $i, j \in \mathbb{N}$, we define $\alpha_{i,j} = \{b_{m,n}\}_{m,n=1}^{\infty}$, where $b_{m,n} = 1$, if $m = i, n = j$ and otherwise $b_{m,n} = 0$. Then $\{\alpha_{i,j}\}_{i,j=1}^{\infty}$ is an orthonormal basis for $l^2(\mathbb{N} \times \mathbb{N})$; it is called the canonical orthonormal basis for $l^2(\mathbb{N} \times \mathbb{N})$.

Theorem 1. *Let $\Lambda = \{\Lambda_i\}_{i=1}^{\infty}$ be a g-frame for H with respect to $\{H_i\}_{i=1}^{\infty}$ and for each $i \in \mathbb{N}$, $\{e_{i,j}\}_{j=1}^{\infty}$ be an orthonormal basis for H_i. The dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$ is precisely the sequence $\Theta = \{\Theta_i\}_{i=1}^{\infty} = \{\phi_i^* V^*\}_{i=1}^{\infty}$, where $V : l^2(\mathbb{N} \times \mathbb{N}) \to H$, is a bounded left inverse of the analysis operator of the frame $\{u_{i,j}\}_{i,j=1}^{\infty}$, and for each $i \in \mathbb{N}$, ϕ_i is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N} \times \mathbb{N})$.*

Proof. Assume that $\{\Theta_i\}_{i=1}^{\infty}$ is a dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$. Then by Theorem 3.1 in [13], for each $i, j \in \mathbb{N}$, $v_{i,j} = \Theta_i^* e_{i,j}$ is a dual frame of $u_{i,j} = \Lambda_i^* e_{i,j}$, defined in (3). By Lemma 5.7.2 in [2],

$$\{v_{i,j}\}_{i,j=1}^{\infty} = \{V \alpha_{i,j}\}_{i,j=1}^{\infty}, \tag{4}$$

where $V : l^2(\mathbb{N}\times\mathbb{N}) \to H$ is a bounded left inverse of T^* (the analysis operator of $\{u_{i,j}\}_{i,j=1}^{\infty}$), and $\{\alpha_{i,j}\}_{i,j=1}^{\infty}$ is the canonical orthonormal basis for $l^2(\mathbb{N}\times\mathbb{N})$. For each $i\in\mathbb{N}$, we define the mapping

$$\phi_i : H_i \to l^2(\mathbb{N}\times\mathbb{N}), \quad \phi_i(\sum_{j=1}^{\infty} c_{i,j}e_{i,j}) = \sum_{j=1}^{\infty} c_{i,j}\alpha_{i,j}. \tag{5}$$

Clearly, the mapping ϕ_i is well defined and is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N}\times\mathbb{N})$. Since for each $i\in\mathbb{N}$, $\{e_{i,j}\}_{j=1}^{\infty}$ is an orthonormal basis for H_i, by (4) and (5), we have

$$\begin{aligned}\Theta_i^*(h) &= \Theta_i^*(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle e_{i,j}) = \sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle v_{i,j} = \sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle V(\alpha_{i,j})\\ &= V(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle \alpha_{i,j}) = V\phi_i(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle e_{i,j}) = V\phi_i(h), \quad i\in\mathbb{N},\ h\in H_i.\end{aligned}$$

So for each $i\in\mathbb{N}$, $\Theta_i = \phi_i^* V^*$.
Now, we show that $\{\Theta_i\}_{i=1}^{\infty} = \{\phi_i^* V^*\}_{i=1}^{\infty}$ is a dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$. We define

$$\widetilde{T} : (\sum_{i=1}^{\infty}\bigoplus H_i)_{l_2} \to H, \quad \widetilde{T}(\{g_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty}\Theta_i^* g_i.$$

Since for each $i\in\mathbb{N}$, $\Theta_i = \phi_i^* V^*$, by (5), we have

$$\begin{aligned}\widetilde{T}(\{g_i\}_{i=1}^{\infty}) &= \sum_{i=1}^{\infty}\Theta_i^* g_i = \sum_{i=1}^{\infty} V\phi_i g_i = V(\sum_{i=1}^{\infty}\phi_i g_i)\\ &= V(\sum_{i=1}^{\infty}\phi_i(\sum_{j=1}^{\infty}\langle g_i, e_{i,j}\rangle e_{i,j})) = V(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle g_i, e_{i,j}\rangle \alpha_{i,j}).\end{aligned} \tag{6}$$

We define the mapping

$$\psi : (\sum_{i=1}^{\infty}\bigoplus H_i)_{l_2} \to l^2(\mathbb{N}\times\mathbb{N}), \quad \psi(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}E_{i,j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\alpha_{i,j}, \tag{7}$$

where $\{E_{i,j}\}_{i,j=1}^{\infty}$ is an orthonormal basis for $(\sum_{i=1}^{\infty}\bigoplus H_i)_{l_2}$. Clearly ψ is a well defined and isometric isomorphism operator. So by (6), (7) and Remark 2, we have

$$\widetilde{T}(\{g_i\}_{i=1}^{\infty}) = V\psi(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle g_i, e_{i,j}\rangle E_{i,j}) = V\psi(\{g_i\}_{i=1}^{\infty}).$$

Therefore, $\tilde{T} = V\psi$. Since V is a bounded left inverse of T^*, V is surjective and hence $\widetilde{T}$ is a well defined, bounded and surjective operator of $(\sum_{i=1}^{\infty} \bigoplus H_i)_{l_2}$ onto H. So, $\Theta = \{\Theta_i\}_{i=1}^{\infty}$ is a g-frame for H with respect to $\{H_i\}_{i=1}^{\infty}$ and $\widetilde{T} = T_\Theta$, where T_Θ is the synthesis operator of $\{\Theta_i\}_{i=1}^{\infty}$.
Now, we prove that $\{\Theta_i\}_{i=1}^{\infty}$ is a dual g-frame of $\{\Lambda_i\}_{i=1}^{\infty}$. Since V is a bounded left inverse of T^*, we have

$$\begin{aligned} f = VT^*f &= V(\{\langle f, u_{i,j}\rangle\}_{i,j=1}^{\infty}) = V(\{\langle f, \Lambda_i^* e_{i,j}\rangle\}_{i,j=1}^{\infty}) \\ &= V(\{\langle \Lambda_i f, e_{i,j}\rangle\}_{i,j=1}^{\infty}), \quad f \in H. \end{aligned} \tag{8}$$

Since $\{\alpha_{i,j}\}_{i,j=1}^{\infty}$ is the canonical orthonormal basis for $l^2(\mathbb{N} \times \mathbb{N})$ and $T_\Theta = V\psi$, by (7), (8) and Remark 2, we have

$$\begin{aligned} f &= T_\Theta \psi^{-1}(\{\langle \Lambda_i f, e_{i,j}\rangle\}_{i,j=1}^{\infty}) = T_\Theta \psi^{-1}(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \Lambda_i f, e_{i,j}\rangle \alpha_{i,j}) \\ &= T_\Theta(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \Lambda_i f, e_{i,j}\rangle E_{i,j}) = T_\Theta(\{\Lambda_i f\}_{i=1}^{\infty}) = T_\Theta T_\Lambda^* f, \quad f \in H. \end{aligned}$$

3. Properties of G-Riesz Bases

Definition 3. *[13] We say that $\{\Lambda_i \in L(H, H_i) : i \in I\}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$ if it is g-complete and there exist constants $A, B > 0$, such that for each finite subset $J \subseteq I$ and $g_i \in H_i$, $i \in J$,*

$$A \sum_{i\in J} \|g_i\|^2 \leq \|\sum_{i\in J} \Lambda_i^* g_i\|^2 \leq B \sum_{i\in J} \|g_i\|^2.$$

We call A and B the g-Riesz basis bounds.

Theorem 2. *[13] A sequence $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$ if and only if there is a g-orthonormal basis $\{Q_i\}_{i\in I}$ for H and a bounded invertible operator T on H such that for each $i \in I$, $\Lambda_i = Q_i T$.*

Corollary 3. *If $\{\Lambda_i = Q_i T \in L(H, H_i) : i \in I\}$ is a g-Riesz basis for H with respect to $\{H_i\}$, then there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H.$$

The largest possible value for the constant A is $\frac{1}{\|T^{-1}\|^2}$ and the smallest possible value for B is $\|T\|^2$.

Theorem 4. *Suppose that for each $i \in I$, $\Lambda_i \in L(H, H_i)$ and $\{e_{i,j}\}_{j\in J_i}$ is an orthonormal basis for H_i. Assume that $\overline{span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$ and for each finite subset $J \subseteq I$,*

$$\|\sum_{i\in J} \Lambda_i^* g_i\|^2 = \sum_{i\in J} \|g_i\|^2, \quad i \in J, \quad g_i \in H_i.$$

Then
(i) $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2$.
(ii) If for each $i \in I$, $(\Lambda_i\Lambda_i^*)^2 = \Lambda_i\Lambda_i^*$, then for all $i \neq j$, $i, j \in I$, $\Lambda_i^*(H_i) \cap \Lambda_j^*(H_j) = \{0\}$.

Proof. (i) By assumption, $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$ and for each $\{g_i\}_{i\in I} \in (\sum_{i\in I} \bigoplus H_i)_{l_2}$,

$$\|\sum_{i\in I} \Lambda_i^* g_i\|^2 = \sum_{i\in I} \|g_i\|^2. \tag{9}$$

Since $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis, by Theorem 2, there exist a g-orthonormal basis $\{Q_i\}_{i\in I}$ and a bounded invertible operator T on H, such that for each $i \in I$, $\Lambda_i = Q_iT$. Since $\{Q_i\}_{i\in I}$ is a g-orthonormal basis, by Remark 1, we have

$$f = \sum_{i\in I} Q_i^* Q_i f, \ f \in H. \tag{10}$$

Hence by (9) and (10),

$$\begin{aligned} \|T^*f\|^2 &= \|T^*(\sum_{i\in I} Q_i^*Q_i f)\|^2 = \|\sum_{i\in I} T^*Q_i^*Q_i f\|^2 \qquad (11) \\ &= \|\sum_{i\in I} \Lambda_i^* Q_i f\|^2 = \sum_{i\in I} \|Q_i f\|^2 = \|f\|^2, \quad f \in H. \end{aligned}$$

Therefore, $\|T^*\| = \|T\| = 1$. Since T^* is invertible, for each $g \in H$, there exists a unique $f \in H$, such that $T^*f = g$. So, by (11), we have

$$\|(T^*)^{-1}g\| = \|f\| = \|T^*f\| = \|g\|.$$

This implies that $\|(T^*)^{-1}\| = 1$ and so $\|T^{-1}\| = 1$. Now, by Corollary 3,

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2. \tag{12}$$

(ii) Let for $i \neq j$, $i, j \in I$, $f \in \Lambda_i^*(H_i) \cap \Lambda_j^*(H_j)$. Then there exist $g_i \in H_i$ and $g_j \in H_j$ such that

$$f = \Lambda_i^* g_i = \Lambda_j^* g_j. \tag{13}$$

By (12) and (13), we have

$$\begin{aligned} \langle \Lambda_i^* g_i, \Lambda_i^* g_i \rangle &= \langle f, f \rangle = \sum_{k \in I} \|\Lambda_k f\|^2 \\ &= \|\Lambda_i \Lambda_i^* g_i\|^2 + \sum_{k \in I, k \neq i} \|\Lambda_k f\|^2. \end{aligned} \tag{14}$$

Since for each $i \in I$, $(\Lambda_i \Lambda_i^*)^2 = \Lambda_i \Lambda_i^*$, by (14),

$$\sum_{k \in I, k \neq i} \|\Lambda_k f\|^2 = 0, \quad i \in I. \tag{15}$$

Similarly, we conclude that

$$\sum_{k \in I, k \neq j} \|\Lambda_k f\|^2 = 0, \quad j \in I. \tag{16}$$

Therefore, by (15) and (16), for each $k \in I$, $\|\Lambda_k f\|^2 = 0$ and so by (12), $f = 0$.

Theorem 5. *[9] Let for each $i \in I$, $\Lambda_i \in L(H, H_i)$. Then the following conditions are equivalent:*
(i) $\{\Lambda_i\}_{i \in I}$ is an exact g-frame for H with respect to $\{H_i\}_{i \in I}$ and

$$\langle \Lambda_{i_1}^* g_{i_1}, \tilde{\Lambda}_{i_2}^* g_{i_2} \rangle = \delta_{i_1, i_2} \langle g_{i_1}, g_{i_2} \rangle, \quad i_1, i_2 \in I, \; g_{i_1} \in H_{i_1}, g_{i_2} \in H_{i_2}, \tag{17}$$

(ii) $\{\Lambda_i\}_{i \in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i \in I}$.

Theorem 6. *Let for each $i \in I$, $\Lambda_i \in L(H, H_i)$. Then we have the followings.*
(i) If $\{\Lambda_i\}_{i \in I}$ is an exact g-frame for H with respect to $\{H_i\}_{i \in I}$, then for each $i \in I$, $\|\tilde{\Lambda}_i \Lambda_i^*\| \geq 1$.
(ii) If $\{\Lambda_i\}_{i \in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i \in I}$, then for each $i \in I$, $\|\tilde{\Lambda}_i \Lambda_i^*\| = 1$.

Proof. (i) Suppose that $\{\Lambda_i\}_{i \in I}$ is an exact g-frame for H. Then By Theorem 3.5 in [13], for each $i \in I$, $I - \tilde{\Lambda}_i \Lambda_i^*$ is not invertible. Therefore, $1 \in \sigma(\tilde{\Lambda}_i \Lambda_i^*)$ and we have $r(\tilde{\Lambda}_i \Lambda_i^*) \geq 1$. Since $\tilde{\Lambda}_i \Lambda_i^*$ is a self adjoint operator on H_i, $r(\tilde{\Lambda}_i \Lambda_i^*) = \|\tilde{\Lambda}_i \Lambda_i^*\|$. So, for each $i \in I$, $\|\tilde{\Lambda}_i \Lambda_i^*\| \geq 1$.
(ii) Suppose that $\{\Lambda_i\}_{i \in I}$ is a g-Riesz basis for H. Then by (17),

$$\begin{aligned} \|\tilde{\Lambda}_i \Lambda_i^*\| &= \sup_{\substack{\|h\|=1 \\ h \in H_i}} |\langle \tilde{\Lambda}_i \Lambda_i^* h, h \rangle| \\ &= \sup_{\substack{\|h\|=1 \\ h \in H_i}} |\langle \Lambda_i^* h, \tilde{\Lambda}_i^* h \rangle| = 1, \quad i \in I. \end{aligned}$$

Definition 4. *[8] Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$. We say that $\{\Lambda_i\}_{i\in I}$ is a Riesz decomposition of H, if for each $f \in H$ there is a unique choice of $f_i \in H_i$ such that $f = \sum_{i\in I} \Lambda_i^* f_i$.*

Definition 5. *[8] A sequence of operators $\{\Lambda_i \in L(H, H_i) : i \in I\}$ is called g-minimal, if for each $j \in I$,*

$$\Lambda_j^*(H_j) \cap \overline{span}\{\Lambda_i^*(H_i)\}_{i\in I, i\neq j} = \{0\}.$$

Definition 6. *[13] The sequences $\{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\{\Theta_i \in L(H, H_i) : i \in I\}$ are said g-biorthogonal if*

$$\langle \Lambda_{i_1}^* f_{i_1}, \Theta_{i_2}^* g_{i_2}\rangle = \delta_{i_1,i_2}\langle f_{i_1}, g_{i_2}\rangle,\ i_1, i_2 \in I,\quad f_{i_1} \in H_{i_1},\ g_{i_2} \in H_{i_2}.$$

Lemma 7. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$. Then the following assertions are equivalent:*
(i) $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$.
(ii) $\{\Lambda_i\}_{i\in I}$ is a Riesz decomposition of H.
(iii) If $\sum_{i\in I} \Lambda_i^* g_i = 0$ for some $\{g_i\}_{i\in I} \in (\sum_{i\in I} \bigoplus H_i)_{l_2}$, then for each $i \in I$, $g_i = 0$.
Moreover if for each $i \in I$, Λ_i is surjective, then the above properties are equivalent to:
(iv) $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ are g-biorthogonal.
(v) $\{\Lambda_i\}_{i\in I}$ has a g-biorthogonal sequence.
(vi) $\{\Lambda_i\}_{i\in I}$ is g-minimal.

Proof. We conclude the equivalence of $(i) \leftrightarrow (ii) \leftrightarrow (iii)$ from Theorem 3.3 in [8].
$(i) \to (iv)$ Suppose that $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$. By Theorem 3.1 in [13], $\{u_{i,j}\}_{i\in I, j\in J_i}$ is a Riesz basis for H and S_Λ (the g-frame operator of $\{\Lambda_i\}_{i\in I}$), is also a frame operator of $\{u_{i,j}\}_{i\in I,j\in J_i}$. Since $\{u_{i,j}\}_{i\in I,j\in J_i}$ and $\{S_\Lambda^{-1} u_{i,j}\}_{i\in I,j\in j_i}$ are biorthogonal, the proof is evident.
$(iv) \to (v)$ It is evident.
$(v) \to (vi)$ Suppose that $\{\Theta_i\}_{i\in I}$ is a g-biorthogonal sequence of $\{\Lambda_i\}_{i\in I}$. Assume that there exist $j \in I$ and $g_j \in H_j$, such that $0 \neq \Lambda_j^*(g_j) \in \overline{span}\{\Lambda_i^*(H_i)\}_{i\in I, i\neq j}$. Then $\langle \Lambda_j^* g_j, \Theta_j^* g_j\rangle = 0$, which is a contradiction.
$(vi) \to (ii)$ Suppose that $f \in H$ and

$$f = \sum_{i\in I} \Lambda_i^* f_i = \sum_{i\in I} \Lambda_i^* g_i,$$

where $f_i, g_i \in H_i$ and $f_j \neq g_j$ for some $j \in I$. Hence

$$\Lambda_j^*(f_j - g_j) = \sum_{i\in I, i\neq j} \Lambda_i^*(g_i - f_i).$$

Since Λ_j is surjective, Λ_j^* is one to one. Hence

$$0 \neq \Lambda_j^*(f_j - g_j) \in \Lambda_j^*(H_j) \cap \overline{span}\{\Lambda_i^*(H_i)\}_{i\in I, i\neq j}.$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is not g-minimal.

Theorem 8. *Suppose that $\{\Lambda_i \in L(H, H_i) : i \in I\}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$ and*

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* g_{i_2}\rangle = \delta_{i_1,i_2}\langle f_{i_1}, g_{i_2}\rangle, \quad i_1, i_2 \in I,\ f_{i_1} \in H_{i_1},\ g_{i_2} \in H_{i_2}. \tag{18}$$

Then $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H.

Proof. Suppose that $f \in H$ and

$$f = \sum_{i\in I} \Lambda_i^* f_i = \sum_{i\in I} \Lambda_i^* g_i,$$

where $f_i, g_i \in H_i$ and $f_j \neq g_j$ for some $j \in I$. Then

$$h = \Lambda_j^*(g_j - f_j) = \sum_{i\in I, i\neq j} \Lambda_i^*(f_i - g_i). \tag{19}$$

By (18) and (19), we have

$$\|h\|^2 = \langle h, h\rangle = \langle \Lambda_j^*(g_j - f_j), \Lambda_j^*(g_j - f_j)\rangle = \|g_j - f_j\|^2,$$

since $f_j \neq g_j$, we conclude that $h \neq 0$.
On the other hand by (18) and (19),

$$\|h\|^2 = \langle h, h\rangle = \langle \Lambda_j^*(g_j - f_j), \sum_{i\in I, i\neq j} \Lambda_i^*(f_i - g_i)\rangle = 0,$$

so, $h = 0$, which is a contradiction. Therefore, $\{\Lambda_i\}_{i\in I}$ is a Riesz decomposition of H and by Lemma 7, $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H.

Corollary 9. *Suppose that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$. Then $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$.*

Theorem 10. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a g-frame for H with respect to $\{H_i\}_{i=1}^{\infty}$. Suppose that there exists $M > 0$, such that for each $i \in \mathbb{N}$ and $g_i \in H_i$, $M\|g_i\| \leq \|\Lambda_i^* g_i\|$. Assume that there exists a constant A such that for all $m, n \in \mathbb{N}$ with $m \leq n$,*

$$\|\sum_{k=1}^{m} \Lambda_k^* g_k\| \leq A\|\sum_{k=1}^{n} \Lambda_k^* g_k\|, \quad g_k \in H_k. \tag{20}$$

Then $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-Riesz Basis for H with respect to $\{H_i\}_{i=1}^{\infty}$.

Proof. For each $i \in \mathbb{N}$ and $m \geq i$, by (20), we have

$$\begin{aligned} M\|g_i\| \leq \|\Lambda_i^* g_i\| &= \|\sum_{k=1}^{i} \Lambda_k^* g_k - \sum_{k=1}^{i-1} \Lambda_k^* g_k\| \leq \|\sum_{k=1}^{i} \Lambda_k^* g_k\| + \|\sum_{k=1}^{i-1} \Lambda_k^* g_k\| \\ &\leq A\|\sum_{k=1}^{m} \Lambda_k^* g_k\| + A\|\sum_{k=1}^{m} \Lambda_k^* g_k\| \\ &= 2A\|\sum_{k=1}^{m} \Lambda_k^* g_k\|. \qquad (21) \end{aligned}$$

So, for each $i \in \mathbb{N}$ and all $n \geq i$ by (21),

$$\|g_i\| \leq \frac{2A}{M}\|\sum_{k=1}^{n} \Lambda_k^* g_k\|. \qquad (22)$$

Now, if $\sum_{k=1}^{\infty} \Lambda_k^* g_k = 0$, then by (22), for each $i \in \mathbb{N}$, $g_i = 0$. Therefore, by Lemma 7, $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-Riesz basis for H.

4. Hilbert-Schmidt Sequences

Definition 7. *[10] Let $u \in L(H, K)$ and suppose that E is an orthonormal basis for H. We define the Hilbert-Schmidt norm of u to be*

$$\|u\|_2 = (\sum_{x \in E} \|u(x)\|^2)^{\frac{1}{2}}.$$

This definition is independent of the choice of basis. If $\|u\|_2 < \infty$, we call u a Hilbert-Schmidt operator.

Definition 8. *[10] Let u be an operator on a Hilbert space H. We define its trace-class norm to be $\|u\|_1 = \||u|^{\frac{1}{2}}\|_2^2$, where $|u| = (u^* u)^{\frac{1}{2}}$. If E is an orthonormal basis for H, then*

$$\|u\|_1 = \sum_{x \in E} \langle |u| x, x \rangle.$$

This definition is independent of the choice of basis. If $\|u\|_1 < \infty$, we call u a trace class operator.

Definition 9. *We say that $\{\Lambda_i \in L(H, H_i) : \ i \in I\}$ is a Hilbert-Schmidt sequence for H with respect to $\{H_i\}_{i \in I}$, if $\{\|\Lambda_i\|_2\}_{i \in I} \in l^2(I)$.*

Lemma 11. *If $\{\Lambda_i\}_{i\in I}$ is a Hilbert-Schmidt sequence for H with respect to $\{H_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in I}$.*

Proof. Suppose that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Then for each $f \in H$, we have

$$\begin{aligned}\sum_{i\in I}\|\Lambda_i f\|^2 &= \sum_{i\in I}\|\Lambda_i(\sum_{k=1}^{\infty}\langle f, e_k\rangle e_k)\|^2 = \sum_{i\in I}\|\sum_{k=1}^{\infty}\langle f, e_k\rangle\Lambda_i e_k\|^2\\ &\le \sum_{i\in I}\sum_{k=1}^{\infty}|\langle f, e_k\rangle|^2\sum_{k=1}^{\infty}\|\Lambda_i e_k\|^2\\ &= \|f\|^2\sum_{i\in I}\sum_{k=1}^{\infty}\|\Lambda_i e_k\|^2\\ &= \|f\|^2\sum_{i\in I}\|\Lambda_i\|_2^2.\end{aligned}$$

Here is an example, which shows that the converse of the above lemma is not true when H is an infinite dimensional Hilbert space.

Example 1. *Assume that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H. A simple calculation shows that $\{\Lambda_i\}_{i=1}^{\infty} = \{e_i \otimes e_i\}_{i=1}^{\infty}$ is a parseval g-frame for H but it is not a Hilbert-Schmidt sequence.*

Lemma 12. *If $\dim H < \infty$, then every g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$ is a Hilbert-Schmidt sequence.*

Proof. Let $\{e_k\}_{k=1}^{n}$ be an orthonormal basis for H. Suppose that $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H. Then there exist an orthonormal basis $\{Q_i\}_{i\in I}$ for H and an invertible operator T on H such that $\Lambda_i = Q_i T$. Therefore, by Theorem 2.4.10 in [10], we have

$$\begin{aligned}\sum_{i\in I}\|\Lambda_i\|_2^2 &= \sum_{i\in I}\|Q_i T\|_2^2 \le \|T\|^2\sum_{i\in I}\|Q_i\|_2^2 = \|T\|^2\sum_{i\in I}\sum_{k=1}^{n}\|Q_i e_k\|^2\\ &= \|T\|^2\sum_{k=1}^{n}\|e_k\|^2 = \|T\|^2\dim H.\end{aligned}$$

The following example shows that the above lemma is not true when H is an infinite dimensional Hilbert space.

Example 2. *Suppose that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H. For each $i \in \mathbb{N}$, we define*

$$\Lambda_i : H \to \mathbb{C}, \quad \Lambda_i f = \langle f, e_i \rangle.$$

A simple calculation shows that for each $\alpha \in \mathbb{C}$, $\Lambda_i^* \alpha = \alpha e_i$. By Lemma 7, $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-Riesz basis for H with respect to $\mathbb{C}$ but it is not a Hilbert-Schmidt sequence.

Theorem 13. *Let $\{\Lambda_i\}_{i\in I}$ be a Hilbert-Schmidt sequence for H with respect to $\{H_i\}_{i\in I}$. Then $S_\Lambda = T_\Lambda T_\Lambda^*$ is a trace class operator.*

Proof. Let $S_\Lambda = U|S_\Lambda|$ be the polar decomposition of S_Λ, where U is a unique partial isometry on H. So $|S_\Lambda| = U^* S_\Lambda$, and we can write

$$|S_\Lambda| = U^* S_\Lambda = U^* T_\Lambda T_\Lambda^* = T_{\Lambda U} T_\Lambda^*,$$

where $T_{\Lambda U}$ is the synthesis operator of $\{\Lambda_i U\}_{i\in I}$. Suppose that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Then

$$\begin{aligned} \|S_\Lambda\|_1 &= \sum_{k=1}^{\infty} \langle |S_\Lambda| e_k, e_k \rangle = \sum_{k=1}^{\infty} \langle T_\Lambda^* e_k, T_{\Lambda U}^* e_k \rangle = \sum_{k=1}^{\infty} \langle \{\Lambda_i e_k\}_{i\in I}, \{\Lambda_i U e_k\}_{i\in I} \rangle \\ &= \sum_{k=1}^{\infty} \sum_{i\in I} \langle \Lambda_i e_k, \Lambda_i U e_k \rangle \le \sum_{k=1}^{\infty} \sum_{i\in I} \|\Lambda_i e_k\| \|\Lambda_i U e_k\| \\ &\le \sum_{i\in I} \Big(\sum_{k=1}^{\infty} \|\Lambda_i e_k\|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} \|\Lambda_i U e_k\|^2\Big)^{\frac{1}{2}} \\ &= \sum_{i\in I} \|\Lambda_i\|_2 \|\Lambda_i U\|_2, \end{aligned} \tag{23}$$

hence by Theorem 2.4.10 in [10] and (23),

$$\|S_\Lambda\|_1 \le \|U\| \sum_{i\in I} \|\Lambda_i\|_2^2.$$

Theorem 14. *Let for each $i \in I$, $H_i \subseteq H$ and $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$ be Hilbert-Schmidt sequences for H with respect to $\{H_i\}_{i\in I}$. Then the following assertions are equivalent:*
(i) $f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \quad f \in H.$
(ii) $f = \sum_{i\in I} \Theta_i^* \Lambda_i f, \quad f \in H.$
(iii) $\langle f, g \rangle = \sum_{i\in I} \langle \Lambda_i f, \Theta_i g \rangle, \quad f, g \in H.$

(iv) $\|f\|^2 = \sum_{i\in I}\langle \Lambda_i f, \Theta_i f\rangle, \quad f \in H.$
(v) For all orthonormal bases $\{e_n\}_{n=1}^{\infty}$ *and* $\{\gamma_m\}_{m=1}^{\infty}$ *for* H*,*

$$\langle e_n, \gamma_m\rangle = \sum_{i\in I}\langle \Lambda_i e_n, \Theta_i \gamma_m\rangle.$$

(vi) For all orthonormal basis $\{e_n\}_{n=1}^{\infty}$ *for* H*,*

$$\langle e_n, e_m\rangle = \sum_{i\in I}\langle \Lambda_i e_n, \Theta_i e_m\rangle.$$

Proof. The equivalence of $(i) \leftrightarrow (ii) \leftrightarrow (iii) \leftrightarrow (iv)$ are evident.
$(v) \to (iii)$ For all $f, g \in H$, we have

$$\begin{aligned} \sum_{i\in I}\langle \Lambda_i f, \Theta_i g\rangle &= \sum_{i\in I}\langle \Lambda_i(\sum_{n=1}^{\infty}\langle f, e_n\rangle e_n), \Theta_i(\sum_{m=1}^{\infty}\langle g, \gamma_m\rangle \gamma_m)\rangle \\ &= \sum_{i\in I}\langle \sum_{n=1}^{\infty}\langle f, e_n\rangle \Lambda_i e_n, \sum_{m=1}^{\infty}\langle g, \gamma_m\rangle \Theta_i \gamma_m\rangle \\ &= \sum_{i\in I}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle f, e_n\rangle\langle \gamma_m, g\rangle\langle \Lambda_i e_n, \Theta_i \gamma_m\rangle. \end{aligned} \tag{24}$$

Since $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$ are Hilbert-Schmidt sequences for H with respect to $\{H_i\}_{i\in I}$, by (24), we have

$$\sum_{i\in I}\langle \Lambda_i f, \Theta_i g\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle f, e_n\rangle\langle \gamma_m, g\rangle\langle e_n, \gamma_m\rangle = \langle f, g\rangle.$$

$(iii) \to (v)$ It is evident.
$(vi) \leftrightarrow (iii)$ It is similar to the proof of $(v) \leftrightarrow (iii)$.

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