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CERTAIN PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce a new class $UT_{q,s}([\alpha_1];\alpha,\beta)$ of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by Dziok-Srivastava operator. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems, the radii of close-to-convexity, starlikeness and convexity and a family of integral operators for functions belonging to the class $UT_{q,s}([\alpha_1];\alpha,\beta)$. We also obtian several results for the modified Hadamard products of functions belonging to this class.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of A which are, respectively, convex and starlike functions of order α, $0 \le \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see [16]).

The Hadamard product (or convolution) $(f*g)(z)$ of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z).$$

For positive real parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0,-1,-2,...; j=1,2,...,s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ is defined by

$$ {}_qF_s(\alpha_1,.....,\alpha_q;\beta_1,.....,\beta_s;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_q)_n}{(\beta_1)_n...(\beta_s)_n n!} z^n $$

$$ (q \leq s+1; s,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \mathbb{N} = \{1,2,.........\};\ z \in U), $$

where $(\theta)_n$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$ (\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0) \\ \theta(\theta+1)....(\theta+n-1) & (n \in \mathbb{N}). \end{cases} $$

For the function $h(\alpha_1,....,\alpha_q;\beta_1,...\beta_s;z) = z{}_qF_s(\alpha_1,.....,\alpha_q;\ \beta_1,.....,\beta_s;\ z)$, the Dziok-Srivastava linear operator (see [5] and [6]) $H_{q,s}(\alpha_1,.....,\alpha_q;\ \beta_1,...$ $...,\beta_s) : A \longrightarrow A$, is defined by the Hadamard product as follows:

$$ \begin{aligned} H_{q,s}(\alpha_1,.....,\alpha_q;\ \beta_1,.....,\beta_s)f(z) &= h(\alpha_1,....,\alpha_q;\beta_1,...\beta_s;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1)\, a_n z^n \quad (z \in U), \end{aligned} \tag{1.2} $$

where

$$ \Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1}.......(\alpha_q)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}\ (n-1)!}. \tag{1.3} $$

For brevity, we write

$$ H_{q,s}(\alpha_1,.....,\alpha_q;\ \beta_1,.....,\beta_s;z)f(z) = H_{q,s}(\alpha_1)f(z). \tag{1.4} $$

For $0 \leq \alpha < 1, \beta \geq 0$ and for all $z \in U$, let $US_{q,s}([\alpha_1];\alpha,\beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$ \operatorname{Re}\left\{ \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - \alpha \right\} > \beta \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right|. \tag{1.5} $$

Denote by T the subclass of A consisting of functions of the form:

$$ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \tag{1.6} $$

which are analytic in U. We define the class $UT_{q,s}([\alpha_1];\alpha,\beta)$ by:

$$UT_{q,s}([\alpha_1];\alpha,\beta) = US_{q,s}([\alpha_1];\alpha,\beta) \cap T. \tag{1.7}$$

We note that for suitable choices of q, s, α and β, we obtain the following subclasses studied by various authors.

(1) For $q = 2, s = 1$ *and* $\alpha_1 = \alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([1];\alpha,\beta)$ reduces to the class $ST(\alpha,\beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{f(z)}{zf'(z)} - \alpha \right\} > \beta \left| \frac{f(z)}{zf'(z)} - 1 \right|,\ 0 \le \alpha < 1, \beta \ge 0, z \in U \right\}$$

and the class $ST(\alpha,0) = ST(\alpha)$ is the class of functions $f(z) \in T$ which satisfy the following condition (see [7] and [17])

$$ST(\alpha) = \mathrm{Re}\left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \quad (0 \le \alpha < 1);$$

(2) For $q = 2, s = 1, \alpha_1 = a\,(a > 0), \alpha_2 = 1$ *and* $\beta_1 = c\,(c > 0)$ in (1.5), the class $UT_{2,1}([a,1;c];\alpha,\beta)$ reduces to the class $\mathcal{L}T(a,c;\alpha,\beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - \alpha \right\} > \beta \left| \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - 1 \right|,\ 0 \le \alpha < 1, \beta \ge 0, a > 0, c > 0, z \in U \right\},$$

where $L(a,c)$ is the Carlson - Shaffer operator (see [2]);

(3) For $q = 2, s = 1, \alpha_1 = \lambda + 1 (\lambda > -1)$ *and* $\alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([\lambda+1];\alpha,\beta)$ reduces to the class $W_\lambda(\alpha,\beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{D^\lambda f(z)}{z(D^\lambda f(z))'} - \alpha \right\} > \beta \left| \frac{D^\lambda f(z)}{z(D^\lambda f(z))'} - 1 \right|,\ 0 \le \alpha < 1, \beta \ge 0, \lambda > -1, z \in U \right\} \ (\textit{ see } [10]\ ),$$

where $D^\lambda (\lambda > -1)$ is the Ruscheweyh derivative operator (see [14]);

(4) For $q = 2, s = 1, \alpha_1 = v + 1\,(v > -1), \alpha_2 = 1$ *and* $\beta_1 = v + 2$ in (1.5), the class $UT_{2,1}([v+1,1;v+2];\alpha,\beta)$ reduces to the class $\zeta T(v;\alpha,\beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{J_v f(z)}{z(J_v f(z))'} - \alpha \right\} > \beta \left| \frac{J_v f(z)}{z(J_v f(z))'} - 1 \right|,\ 0 \le \alpha < 1, \beta \ge 0, v > -1, z \in U \right\},$$

where $J_v f(z)$ is the generalized Bernardi - Libera - Livingston operator (see [1], [8] and [9]);

(5) For $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = 2 - \mu\,(\mu \neq 2, 3, ...)$ in (1.5), the class $UT_{2,1}([2, 1; 2 - \mu]; \alpha, \beta)$ reduces to the class $\mathcal{F}T(\mu; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re}\left\{ \frac{\Omega_z^\mu f(z)}{z(\Omega_z^\mu f(z))'} - \alpha \right\} > \beta \left| \frac{\Omega_z^\mu f(z)}{z(\Omega_z^\mu f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \right. \\ \left. \geq 0, \mu \neq 2, 3, ..., z \in U \right\},$$

where $\Omega_z^\mu f(z)$ is the Srivastava - Owa fractional derivative operator (see [12] and [13]);

(6) For $q = 2, s = 1, \alpha_1 = \mu\,(\mu > 0)\,, \alpha_2 = 1$ *and* $\beta_1 = \lambda + 1\,(\lambda > -1)$ in (1.5), the class $UT_{2,1}([\mu, 1; \lambda + 1]; \alpha, \beta)$ reduces to the class $\pounds T(\mu, \lambda; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re}\left\{ \frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - \alpha \right\} > \beta \left| \frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \right. \\ \left. \beta \geq 0, \mu > 0, \lambda > -1, z \in U \right\},$$

where $I_{\lambda,\mu} f(z)$ is the Choi-Saigo-Srivastava operator (see [4]);

(7) For $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = k + 1\,(k > -1)$ in (1.5), the class $UT_{2,1}([2, 1; k + 1]; \alpha, \beta)$ reduces to the class $AT(k; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re}\left\{ \frac{I_k f(z)}{z(I_k f(z))'} - \alpha \right\} > \beta \left| \frac{I_k f(z)}{z(I_k f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \right. \\ \left. \beta \geq 0, k > -1, z \in U \right\},$$

where $I_k f(z)$ is the Noor integral operator (see [11]);

(8) For $q = 2, s = 1, \alpha_1 = c\,(c > 0)\,, \alpha_2 = \lambda + 1\,(\lambda > -1)$ *and* $\beta_1 = a\,(a > 0)$ in (1.5), the class $UT_{2,1}([c, \lambda + 1; a]; \alpha, \beta)$ reduces to the class $\digamma T(c,$
$a, \lambda; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re}\left\{ \frac{I^\lambda(a, c) f(z)}{z(I^\lambda(a, c) f(z))'} - \alpha \right\} > \beta \left| \frac{I^\lambda(a, c) f(z)}{z(I^\lambda(a, c) f(z))'} - 1 \right|,\ 0 \leq \right. \\ \left. \alpha < 1, \beta \geq 0, c > 0, \lambda > -1, a > 0, z \in U \right\},$$

where $I^\lambda(a, c) f(z)$ is the Cho-Kwon-Srivastava operator (see [3]).

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ are positive real numbers, $0 \leq \alpha < 1,\ \beta \geq 0,\ n \geq 2,\ z \in U$ *and* $\Psi_n(\alpha_1)$ is defined by (1.3).

Theorem 1. *A function $f(z)$ of the form (1.6) is in the class $UT_{q,s}([\alpha_1];\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)\, a_n \leq 1-\alpha. \tag{2.1}$$

Proof. Suppose that (2.1) is true. Since

$$\frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{1-\alpha} - n\Psi_n(\alpha_1) = \frac{(n-1)(1+\beta)\Psi_n(\alpha_1)}{1-\alpha} > 0,$$

we deduce

$$\sum_{n=2}^{\infty} n\Psi_n(\alpha_1)\, a_n < \sum_{n=2}^{\infty}\frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{1-\alpha} a_n \leq 1.$$

It suffices to show that

$$\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right| - \operatorname{Re}\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right) \leq 1-\alpha,$$

we have

$$\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right| - \operatorname{Re}\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right)$$

$$\begin{aligned} &\leq (1+\beta)\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right| \\ &\leq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)\Psi_n(\alpha_1)\, a_n}{1-\sum_{n=2}^{\infty} n\Psi_n(\alpha_1)\, a_n}, \end{aligned}$$

which yields

$$(1-\alpha)-(1+\beta)\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right|$$
$$> \frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)\,a_n}{1-\sum\limits_{n=2}^{\infty}n\Psi_n(\alpha_1)\,a_n} \geq 0. \tag{2.2}$$

This completes the proof of Theorem 1.

Unfortunately, the converse of the above Theorem 1 is not true. So we define the subclass $T_{q,s}([\alpha_1];\alpha,\beta)$ of $UT_{q,s}([\alpha_1];\alpha,\beta)$ consisting of functions $f(z)$ which satisfy (2.1).

Remark 1. Putting $q=2, s=1, \beta=0$ and $\alpha_1=\alpha_2=\beta_1=1$, in Theorem 1 reduces to the result obtained by Yamakawa [17, Lemma 2.1, with $n=p=1$].

Corollary 1. *Let the function* $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1];$ $\alpha,\beta)$, *then*

$$a_n \leq \frac{(1-\alpha)}{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}\ (n\geq 2). \tag{2.3}$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\alpha)}{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}z^n\ (n\geq 2). \tag{2.4}$$

Putting $q=2, s=1, \alpha_1=\lambda+1(\lambda>-1)$ *and* $\alpha_2=\beta_1=1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *A function* $f(z)$ *of the form (1.6) is in the class* $W_\lambda(\alpha,\beta)$ *if*

$$\sum_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\frac{(\lambda+1)_{n-1}}{(n-1)!}a_n \leq 1-\alpha.$$

Remark 2. The result in Corollary 2 correct the result obtained by Najafzadeh and Kulkarni [10, Lemma 1.1].

3. Distortion theorems

Theorem 2. *Let the function $f(z)$ defined by (1.6) belong to the class $T_{q,s}([\alpha_1];\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$r-\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r^2\leq|f(z)|\leq r+\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r^2, \tag{3.1}$$

provided $\Psi_n(\alpha_1)\geq\Psi_2(\alpha_1)$ $(n\geq 2)$. The result is sharp with equlaity for the function $f(z)$ defined by

$$f(z)=z-\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}z^2 \tag{3.2}$$

at $z=r$ and $z=re^{i(2n+1)\pi}$ $(n\in\mathbb{N})$.

Proof. We have

$$|f(z)|\leq r+\sum_{n=2}^{\infty}a_nr^n\leq r+r^2\sum_{n=2}^{\infty}a_n. \tag{3.3}$$

Since for $n\geq 2$, we have

$$(3-2\alpha+\beta)\Psi_2(\alpha_1)\leq[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1),$$

then (2.1) yields

$$(3-2\alpha+\beta)\Psi_2(\alpha_1)\sum_{n=2}^{\infty}a_n\leq\sum_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)\,a_n\leq(1-\alpha) \tag{3.4}$$

or

$$\sum_{n=2}^{\infty}a_n\leq\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}. \tag{3.5}$$

From (3.5) and (3.3) we have

$$|f(z)|\leq r+\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r^2$$

and similarly, we have

$$|f(z)|\geq r-\frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r^2.$$

This completes the proof of Theorem 2.

Theorem 3. *Let the function $f(z)$ defined by (1.6) belong to the class $T_{q,s}([\alpha_1];\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$1-\tfrac{2(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r \le \left|f'(z)\right| \le 1+\tfrac{2(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}r, \tag{3.6}$$

provided $\Psi_n(\alpha_1) \ge \Psi_2(\alpha_1)$ $(n \ge 2)$. The result is sharp for the function $f(z)$ given by (3.2) at $z=r$ and $z=re^{i(2n+1)\pi}$ $(n \in \mathbb{N})$.

Proof. For a function $f(z) \in UT_{q,s}([\alpha_1];\alpha,\beta)$, it follows from (2.2) and (3.5) that

$$\sum_{n=2}^{\infty} na_n \le \frac{2(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)}.$$

4. Extreme points

Theorem 4. *The class $T_{q,s}([\alpha_1];\alpha,\beta)$ is closed under convex linear combinations.*

Proof. Let $f_j(z) \in T_{q,s}([\alpha_1];\alpha,\beta)$ $(j=1,2)$, where

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \ge 0;\ j=1,2). \tag{4.1}$$

Then it is sufficient to prove that the function $h(z)$ given by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \le \mu \le 1)$$

is also in the class $T_{q,s}([\alpha_1];\alpha,\beta)$. For $0 \le \mu \le 1$

$$h(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n$$

and with the aid of Theorem 1, we have

$$\sum_{n=2}^{\infty} [2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1) \cdot [\mu a_{n,1} + (1-\mu) a_{n,2}]$$

$$\le \mu(1-\alpha) + (1-\mu)(1-\alpha) = 1-\alpha,$$

which implies that $h(z) \in T_{q,s}([\alpha_1];\alpha,\beta)$. This completes the proof of Theorem 4. As a consequence of Theorem 4, there exist extreme points of the class $T_{q,s}([\alpha_1];\alpha,\beta)$, which are given by:

Theorem 5. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{(1-\alpha)}{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)} z^n \quad (n \geq 2).$$

Then $f(z)$ *is in the class* $T_{q,s}([\alpha_1];\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{4.2}$$

where $\mu_n \geq 0$ $(n \geq 1)$ *and* $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{(1-\alpha)}{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)} \mu_n z^n. \end{aligned}$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)} \frac{(1-\alpha)}{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)} \mu_n$$

$$= \sum_{n=2}^{\infty} \mu_n = (1-\mu_1) \leq 1. \tag{4.3}$$

So, by Theorem 1, we have $f(z) \in T_{q,s}([\alpha_1];\alpha,\beta)$.
Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_{q,s}([\alpha_1];\alpha,\beta)$. Then a_n are given by (2.3). Setting

$$\mu_n = \frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)} a_n \tag{4.4}$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n,$$

we can see that $f(z)$ can be expressed in the form (4.2). This completes the proof of Theorem 5.

Corollary 3. *The extreme points of the class* $T_{q,s}([\alpha_1];\alpha,\beta)$ *are the functions* $f_1(z)=z$ *and*

$$f_n(z)=z-\frac{(1-\alpha)}{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}z^n (n\geq 2).$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 6. *Let the function* $f(z)$ *defined by (1.6) be in the class* $T_{q,s}([\alpha_1];$ $\alpha,\beta)$. *Then* $f(z)$ *is close-to-convex of order* ρ $(0\leq\rho<1)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{n\geq 2}\left\{\frac{(1-\rho)\,[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{n(1-\alpha)}\right\}^{\frac{1}{n-1}}. \qquad (5.1)$$

The result is sharp, the extremal function being given by (2.4).

Proof. We must show that

$$\left|f'(z)-1\right|\leq 1-\rho \qquad \text{for } |z|<r_1,$$

where r_1 is given by (5.1). Indeed we find from the definition (1.6) that

$$\left|f'(z)-1\right|\leq\sum_{n=2}^{\infty}na_n\,|z|^{n-1}.$$

Thus

$$\left|f'(z)-1\right|\leq 1-\rho,$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right)a_n\,|z|^{n-1}\leq 1. \qquad (5.2)$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1}\leq\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)},$$

that is, if

$$|z|\leq\left\{\frac{(1-\rho)[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{n(1-\alpha)}\right\}^{\frac{1}{n-1}} (n\geq 2). \qquad (5.3)$$

Theorem 6 follows easily from (5.3).

Theorem 7. *Let the function $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1];\alpha,\beta)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_{n\geq 2}\left\{\frac{(1-\rho)\,[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(n-\rho)\,(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).
Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \quad (|z|<r_2),$$

where r_2 is given by (5.4). Indeed we find, again from the definition (1.6) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n\,|z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho,$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n\,|z|^{n-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \leq \frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}$$

that is, if

$$|z| \leq \left\{\frac{(1-\rho)\,[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(n-\rho)\,(1-\alpha)}\right\}^{\frac{1}{n-1}} \quad (n\geq 2). \tag{5.6}$$

Theorem 7 follows easily from (5.6).

Similarly, we can prove the following theorem.

Theorem 8. *Let the functions $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1];\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_{n\geq 2} \left\{ \frac{(1-\rho)\left[2n - n(\alpha-\beta) - (\beta+1)\right]\Psi_n(\alpha_1)}{n(n-\rho)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

6. A family of integral operators

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1];\alpha,\beta)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt \quad (c > -1) \tag{6.1}$$

also belongs to the class $T_{q,s}([\alpha_1];\alpha,\beta)$.

Proof. Let the function $f(z)$ be defined by (1.6). Then from the representation (6.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} d_n z^n,$$

where

$$d_n = \left(\frac{c+1}{c+n}\right) a_n.$$

Therefore, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} [2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)\, d_n \\ = & \sum_{k=2}^{\infty} [2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1) \left(\frac{c+1}{c+n}\right) a_n \\ \leq & \sum_{n=2}^{\infty} [2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)\, a_n \leq (1-\alpha) \end{aligned}$$

since $f(z) \in T_{q,s}([\alpha_1];\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in T_{q,s}([\alpha_1];\alpha,\beta)$. This completes the proof of Theorem 9.

Theorem 10. *Let the function* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0)$ *be in the class* $T_{q,s}([\alpha_1];\alpha,\beta)$ *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *given by* (6.1) *is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_{n\geq 2}\left\{\frac{(c+1)\left[2n - n(\alpha-\beta) - (\beta+1)\right]\Psi_n(\alpha_1)}{n(c+n)(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{6.2}$$

The result is sharp.

Proof. From (6.1) we have

$$\begin{aligned} f(z) &= \frac{z^{1-c}}{c+1}\left(z^c F(z)\right)' \\ &= z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k z^k. \end{aligned}$$

To prove the assertion of the theorem, it suffices to show that

$$\left|f'(z) - 1\right| < 1 \text{ for } |z| < R^*,$$

where R^* is defined by (6.2). Now

$$\begin{aligned} \left|f'(z) - 1\right| &= \left|-\sum_{n=2}^{\infty} n\left(\frac{c+n}{c+1}\right) a_n z^{n-1}\right| \\ &\leq \sum_{n=2}^{\infty} n\left(\frac{c+n}{c+1}\right) a_n |z|^{n-1}. \end{aligned}$$

Thus

$$\left|f'(z) - 1\right| < 1 \text{ if } \sum_{n=2}^{\infty} n\left(\frac{c+n}{c+1}\right) a_n |z|^{n-1} < 1.$$

But Theorem 1 confirms that

$$\sum_{n=2}^{\infty}\frac{\left[2n - n(\alpha-\beta) - (\beta+1)\right]\Psi_n(\alpha_1)}{(1-\alpha)} a_n \leq 1. \tag{6.3}$$

Thus (6.3) will be satisfied if

$$n\left(\frac{c+n}{c+1}\right)|z|^{n-1} \leq \sum_{n=2}^{\infty}\frac{\left[2n - n(\alpha-\beta) - (\beta+1)\right]\Psi_n(\alpha_1)}{(1-\alpha)} \;(n \geq 2),$$

that is, if

$$|z| \leq \left\{ \frac{(c+1)\left[2n - n(\alpha - \beta) - (\beta+1)\right]\Psi_n(\alpha_1)}{n(c+n)(1-\alpha)} \right\}^{\frac{1}{n-1}} (n \geq 2). \tag{6.4}$$

The required result follows now from (6.4).
Finally, the result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{(c+1)(1-\alpha)}{(c+n)\left[2n - n(\alpha-\beta) - (\beta+1)\right]\Psi_n(\alpha_1)} z^n \; (n \geq 2; c > -1). \tag{6.5}$$

7. Modified Hadamard products

Let the functions $f_j(z)(j = 1, 2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z). \tag{7.1}$$

Theorem 11. *Let each of the functions $f_j(z)(j = 1, 2)$ defined by (4.1) be in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. If the sequence $\{\delta_n(\alpha, \beta)\}$ $(n \geq 2)$,* where

$$\delta_n(\alpha, \beta) = \{[2n - n(\alpha - \beta) - (\beta+1)]\Psi_n(\alpha_1)\} \tag{7.2}$$

is non-decreasing, then $(f_1 * f_2)(z) \in T_{q,s}([\alpha_1]; \eta(q, s, \Psi_2(\alpha_1), \alpha, \beta), \beta)$ where η is given by

$$\eta(q, s, \Psi_2(\alpha_1), \alpha, \beta) = 1 - \frac{(1+\beta)(1-\alpha)^2}{(3 - 2\alpha + \beta)^2 \Psi_2(\alpha_1) - 2(1-\alpha)^2}. \tag{7.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Sliverman [15], we need to fined the largest $\eta = \eta(q, s, \Psi_2(\alpha_1), \alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} \frac{[2n - n(\eta - \beta) - (\beta+1)]\Psi_n(\alpha_1)}{(1-\eta)} a_{n,1} a_{n,2} \leq 1. \tag{7.4}$$

Since

$$\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)} a_{n,1} \leq 1, \tag{7.5}$$

and

$$\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1-\alpha)} a_{n,2} \leq 1, \tag{7.6}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1-\alpha)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[2n - n(\eta - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1-\eta)} a_{n,1}a_{n,2} \\ \leq &\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1-\alpha)} \sqrt{a_{n,1}a_{n,2}}, \end{aligned}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1-\eta)[2n - n(\alpha - \beta) - (\beta + 1)]}{(1-\alpha)[2n - n(\eta - \beta) - (\beta + 1)]}.$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1-\alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}.$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} \leq \frac{(1-\eta)[2n - n(\alpha - \beta) - (\beta + 1)]}{(1-\alpha)[2n - n(\eta - \beta) - (\beta + 1)]},$$

or, equivalently, that

$$\eta \leq 1 - \frac{(n-1)(1+\beta)(1-\alpha)^2}{[2n - n(\alpha - \beta) - (\beta + 1)]^2\Psi_n(\alpha_1) - n(1-\alpha)^2}.$$

Since

$$\varphi(n) = 1 - \frac{(n-1)(1+\beta)(1-\alpha)^2}{[2n - n(\alpha - \beta) - (\beta + 1)]^2\Psi_n(\alpha_1) - n(1-\alpha)^2} \tag{7.8}$$

is an increasing function of n $(n \geq 2)$, letting $n = 2$ in (7.8), we obtain

$$\eta \leq \varphi(2) = 1 - \frac{(1+\beta)(1-\alpha)^2}{(3 - 2\alpha + \beta)^2\,\Psi_2(\alpha_1) - 2(1-\alpha)^2}, \tag{7.9}$$

which proves the main assertion of Theorem 11.
Finally, by taking the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z - \frac{(1-\alpha)}{(3 - 2\alpha + \beta)\,\Psi_2(\alpha_1)} z^2 \quad (j = 1, 2), \tag{7.10}$$

we can see that the result is sharp.

Remark 3. Putting $q=2, s=1, \beta=0$ and $\alpha_1=\alpha_2=\beta_1=1$, in Theorem 11, we will obtain the result obtained by Kang et al. [7, Corollary 1, with $n=p=1$ and $m=2$].

Theorem 12. *Let the functions $f_j(z)(j=1,2)$ defined by (4.1) be in the class $T_{q,s}([\alpha_1];\alpha,\beta)$ if the sequence $\{\delta_n(\alpha,\beta)\}$ $(n\geq 2)$* defined by *(7.2) is non-decreasing, then the function*

$$g(z)=z-\sum_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{7.11}$$

belongs to the class $T_{q,s}([\alpha_1];\xi(q,s,\Psi_2(\alpha_1),\alpha,\beta),\beta)$, where

$$\xi(q,s,\Psi_2(\alpha_1),\alpha,\beta)=1-\frac{2(1+\beta)(1-\alpha)^2}{(3-2\alpha+\beta)^2\Psi_2(\alpha_1)-4(1-\alpha)^2}. \tag{7.12}$$

The result is sharp for the functions $f_j(z)$ given by (7.10).
Proof. By virture of Theorem 1, we obtain

$$\sum_{n=2}^{\infty}\left\{\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}\right\}^2 a_{n,1}^2$$

$$\leq\left\{\sum_{n=2}^{\infty}\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}a_{n,1}\right\}^2\leq 1, \tag{7.13}$$

and

$$\sum_{n=2}^{\infty}\left\{\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}\right\}^2 a_{n,2}^2$$

$$\leq\left\{\sum_{n=2}^{\infty}\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}a_{n,2}\right\}^2\leq 1, \tag{7.14}$$

it follows from (7.13) and (7.14) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}\right\}^2(a_{n,1}^2+a_{n,2}^2)\leq 1.$$

Therefore, we need to find the largest $\xi(q,s,\Psi_2(\alpha_1),\alpha,\beta)$ such that

$$\frac{[2n-n(\zeta-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\zeta)}\leq\frac{1}{2}\left\{\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}\right\}^2 \ (n\geq 2),$$

that is,

$$\zeta \leq 1 - \frac{2(n-1)(1+\beta)(1-\alpha)^2}{[2n - n(\alpha-\beta) - (\beta+1)]^2 \Psi_n(\alpha_1) - 2n(1-\alpha)^2} \ (n \geq 2).$$

Since

$$B(n) = 1 - \frac{2(1-n)(1+\beta)(1-\alpha)^2}{[(1+\beta) - n(\alpha-\beta)]^2 \Psi_n(\alpha_1) - 2n(1-\alpha)^2}$$

is an increasing function of $n(n \geq 2)$, we readily have

$$\zeta \leq B(2) = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(3-2\alpha+\beta)^2 \Psi_2(\alpha_1) - 4(1-\alpha)^2}.$$

This completes the proof of Theorem 12.

Remark 4. Specializing the parameters $q, s, \alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$, in the above results, we obtain the corresponding results for the corresponding classes defined in the introduction.

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