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# SOME RESULTS BETWEEN CERTAIN INEQUALITIES AND $\lambda$-SPIRALLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract. In the light of a novel proof technique, several comprehensive relations between certain complex inequalities and $\lambda$-spirallike functions of complex order are first derived, and some useful consequences of them, which are related to analytic and geometric function theory, are then pointed out.

*Keywords*: Unit open disk, analytic, univalent, inequaliteis in the complex plane, $\lambda$-spirallike type function, $\lambda$-Robertson type function, one variable complex function with complex order.

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## 1. Introduction, Definitions And Motivation

Here and throughout this paper, let us denote $\mathbb{U}$, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$, the unit open disk, the set of complex numbers, the set of real numbers and the set of natural numbers, respectively. Also, let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \ \big(n \in \mathbb{N}\big),$$

which are *analytic* and *univalent* in the complex domain $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

The aim of this investigation to reveal some general results (dealing with the functions $f(z)$ belonging to the general class $\mathcal{A}$) between certain complex inequalities and $\lambda$-spirallike complex type functions of complex order and to point a number of them which will be useful for analytic and geometric function theory (see, [6]) out, as the consequences of the main results which will be presented in the third section. For the proofs of all results, a novel proof technique, which is the assertion (below) obtained by Nunokawa in [12], are also used. In the literature, there are several theorems on which used different methods for their proofs. For those, one may focus on certain papers which are given by the references in [1]-[7] and also [9]-[17].

Before stating and then proving the main results given in the second section, we need here to recall the following lemma due to Nunokawa [12].

**Lemma 1.** *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0)=1$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\Re e\Big(p(z)\Big) > 0 \quad \Big(|z| < |z_0|\Big) \quad , \quad \Re e\Big(p(z_0)\Big) = 0 \quad and \quad p(z_0) \neq 0,$$

*then*

$$p(z_0) = ia \quad and \quad \frac{zp'(z_0)}{p(z_0)} = ik\left(a + \frac{1}{a}\right),$$

*where $a \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$ and $k \geq 1$.*

## 2. The Main Results

Now, with the help of the lemma above, we can prove the following result.

**Theorem 1.** *Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $w \in \mathbb{C}^*$, $b \in \mathbb{C}^*$ and also let $\lambda \in \mathbb{R}^*$ with $|\lambda| < \pi/2$. If $f(z)$ satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\big(f'(z)-1\big)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Im m\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^2} & if \quad \Im m\left(b\overline{w}e^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \quad \Im m\left(b\overline{w}e^{-i\lambda}\right) = 0 \end{cases} \tag{1}$$

*and*

$$\left|\Im m\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\big(f'(z)-1\big)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Re e\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^2} & if \quad \Re e\left(b\overline{w}e^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \quad \Re e\left(b\overline{w}e^{-i\lambda}\right) = 0 \end{cases} , \tag{2}$$

*then*

$$\Re e\left[\left(1+\frac{e^{i\lambda}}{b\cos\lambda}\big(f'(z)-1\big)\right)^w\right] > 0 \quad \Big(z \in \mathbb{U};\ w, b \in \mathbb{C}^*; \lambda \in \mathbb{R}^*\ \big(|\lambda| < \pi/2\big)\Big),$$

*where the value of the power above is taken to be as its principal value.*

*Proof.* Let us define $p(z)$ by

$$p(z) = \left(1+\frac{e^{i\lambda}}{b\cos\lambda}\Big(f'(z)-1\Big)\right)^w \quad \Big(z \in \mathbb{U};\ w, b \in \mathbb{C}^*;\ |\lambda| < \pi/2\Big).$$

It is clear that $p(z)$ is an analytic function in $\mathbb{U}$ and also satisfies $p(0)=1$, and it follows that

$$\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)}=\frac{be^{-i\lambda}\cos\lambda}{w}\cdot\frac{zp'(z)}{p(z)}\quad\Big(z\in\mathbb{U};\ w,b\in\mathbb{C}^*;\ |\lambda|<\pi/2\Big).\tag{3}$$

Suppose now that there exists a point $z_0\in\mathbb{U}$ such that

$$\Re e\Big(p(z)\Big)>0\quad\Big(|z|<|z_0|\Big),\ \ \Re e\Big(p(z_0)\Big)=0\ \text{ and }\ p(z_0)\neq 0.$$

From the Lemma 1, we then obtain that

$$p(z_0)=ia\quad\text{and}\quad\frac{zp'(z_0)}{p(z_0)}=ik\left(a+\frac{1}{a}\right)\quad\Big(a\in\mathbb{R}^*;k\geq 1\Big).\tag{4}$$

With the help of (3) and (4), we also get that

$$\Re e\left(\frac{z_0f''(z_0)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z_0)-1\right)}\right)=\Re e\left(\frac{be^{-i\lambda}\cos\lambda}{w}\cdot\frac{zp'(z)}{p(z)}\right)$$

$$=-\frac{k\cos\lambda}{2\left|w\right|^2}\,\Im m\left(b\overline{w}e^{-i\lambda}\right)\psi(a)\tag{5}$$

and

$$\Im m\left(\frac{z_0f''(z_0)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z_0)-1\right)}\right)=\Im m\left(\frac{be^{-i\lambda}\cos\lambda}{w}\cdot\frac{zp'(z)}{p(z)}\right)$$

$$=\frac{k\cos\lambda}{2\left|w\right|^2}\,\Re e\left(b\overline{w}e^{-i\lambda}\right)\psi(a),\tag{6}$$

where

$$\psi(a)=a+\frac{1}{a}\quad(a\in\mathbb{R}^*).$$

Since the function $\psi(a)=\left|a+\frac{1}{a}\right|$ has minimal value 2, the equalities in (5) and (6) immediately yield that

$$\left|\Re e\left(\frac{z_0f''(z_0)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z_0)-1\right)}\right)\right|\begin{cases}\geq\frac{\cos\lambda\left|\Im m\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^2} & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)\neq 0\\ =0 & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)=0\end{cases}$$

and

$$\left|\Im m\left(\frac{z_0f''(z_0)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z_0)-1\right)}\right)\right|\begin{cases}\geq\frac{\cos\lambda\left|\Re e\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^2} & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)\neq 0\\ =0 & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)=0\end{cases},$$

respectively. But, the cases of these inequalities are contradictions with the cases of the inequalities in (1) and (2), respectively. Hence, $\Re e\{p(z)\} > 0$ for all $z \in \mathbb{U}$, therefore, we have

$$\Re e\left[\left(1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)\right)^{w}\right] > 0 \quad \left(z\in\mathbb{U}; w, b\in\mathbb{C}^{*}; \lambda\in\mathbb{R}^{*}\ \left(|\lambda|<\pi/2\right)\right).$$

By use of the same technique that we used in the proof of the Theorem 1, the following results, which are Theorems 2 and 3 below, can be easily proven. Therefore, their proofs are here omitted.

**Theorem 2.** *Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $w \in \mathbb{C}^{*}$, $b \in \mathbb{C}^{*}$ and also let $\lambda \in \mathbb{R}^{*}$ with $|\lambda| < \pi/2$. If $f(z)$ satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{f'(z)-\frac{f(z)}{z}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Im m\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^{2}} & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)\neq 0 \\ \neq 0 & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)=0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{f'(z)-\frac{f(z)}{z}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Re e\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^{2}} & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)\neq 0 \\ \neq 0 & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)=0 \end{cases},$$

*then*

$$\Re e\left[\left(1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)\right)^{w}\right] > 0 \quad \left(z\in\mathbb{U};\ w, b\in\mathbb{C}^{*};\ \lambda\in\mathbb{R}^{*}\ \left(|\lambda|<\pi/2\right)\right),$$

*where the value of the complex power is chosen to be as its principal value.*

**Theorem 3.** *Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $w \in \mathbb{C}^{*}$, $b \in \mathbb{C}^{*}$ and also let $\lambda \in \mathbb{R}^{*}$ with $|\lambda| < \pi/2$. Then, if the function $\mathcal{F}(z)$, which is defined by*

$$\mathcal{F}(z) = (1-\mu)f(z) + \mu z f'(z) \quad (f(z)\in\mathcal{A};\ 0\leq\mu\leq 1),$$

*satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)}-\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{2}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Im m\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^{2}} & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)\neq 0 \\ \neq 0 & if\ \ \Im m\left(b\overline{w}e^{-i\lambda}\right)=0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)}-\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{2}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)}\right)\right| \begin{cases} < \frac{\cos\lambda\left|\Re e\left(b\overline{w}e^{-i\lambda}\right)\right|}{|w|^{2}} & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)\neq 0 \\ \neq 0 & if\ \ \Re e\left(b\overline{w}e^{-i\lambda}\right)=0 \end{cases},$$

*then*

$$\Re e\left\{\left[1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)\right]^{w}\right\}>0$$

$$\Big(0\leq\mu\leq1;\ w,b\in\mathbb{C}^*;\ \lambda\in\mathbb{R}^*\ \big(|\lambda|<\pi/2\big);\ z\in\mathbb{U}\Big),$$

*where each one of the values of the exponential forms above is considered to be as its principal value.*

## 3. Some Consequences Of The Main Results

In this section, to center on some applications of the results produced in the second section, we need to recall some functions classes which have important role in analytic and geometric function theory (cf., e.g., [6]). For those, a function $f(z)\in\mathcal{A}$ is said to be $\lambda$-spirallike function of complex order in $\mathbb{U}$, denoted by $\mathcal{S}^\lambda(b)$ if and only if

$$\Re e\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0\quad\Big(z\in\mathbb{U}\Big),$$

for some real numbers $b\in\mathbb{C}^*:=\mathbb{C}\diagdown\{0\}$ and $\lambda\in\mathbb{R}$ with $|\lambda|<\pi/2$. The class $\mathcal{S}^\lambda(b)$ was introduced and studied by Al-Oboudi and Haidan [1]. A function $f(z)\in\mathcal{A}$ is also said to be $\lambda$- Robertson function of complex order in $\mathbb{U}$ if and only if

$$\Re e\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0\quad\Big(z\in\mathbb{U}\Big),$$

for some real numbers $b\in\mathbb{C}^*$ and and $\lambda\in\mathbb{R}$ with $|\lambda|<\pi/2$. We denote this class by $\mathcal{C}^\lambda(b)$ (see, [2]). From the definitions just above, it is clear that

$$f(z)\in\mathcal{C}^\lambda(b)\quad\Longleftrightarrow\quad zf'(z)\in\mathcal{S}^\lambda(b).$$

Let $\mathcal{R}^\lambda(b)$ denote the class of functions $f(z)\in\mathcal{A}$ such that

$$\Re e\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\Big(f'(z)-1\Big)\right\}>0\quad\Big(z\in\mathbb{U};\,b\in\mathbb{C}^*\Big),$$

also let $\mathcal{D}^\lambda(b)$ denote the class of functions $f(z)\in\mathcal{A}$ such that

$$\Re e\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)\right\}>0\quad\Big(z\in\mathbb{U};b\in\mathbb{C}^*\Big).$$

Noting that the above functions classes include several subclasses which have important role in the analytic and geometric function theory. From this reason, we want to state some of them:

- $\mathcal{S}^\lambda(1) =: \mathcal{S}^\lambda$ $(|\lambda| < \pi/2)$ is known the $\lambda$-spirallike univalent functions class and was defined by Spacek [15], $\mathcal{S}^0(b) =: \mathcal{S}(b)$ $(b \in \mathbb{C}^*)$ is said to be the starlike functions class of complex order and was studied by Nasr and Aouf [10], $\mathcal{S}^\lambda(1-\alpha) =: \mathcal{S}^\lambda(\alpha)$ $(0 \leq \alpha < 1)$ is known the $\lambda$-spirallike functions class of order $\alpha$ and was studied by Libera [9] and $\mathcal{S}^0(1-\alpha) =: \mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$ is said to be the starlike functions class of order $\alpha$ $(0 \leq \alpha < 1)$ and was studied by Robertson [13].

- $\mathcal{C}^\lambda(1) =: \mathcal{C}^\lambda$ $(|\lambda| < \pi/2)$ is known the $\lambda$-Robertson type functions class and was first studied by Robertson [14], $\mathcal{C}^0(b) =: \mathcal{C}(b)$ is called the convex functions class of complex order and was studied by Waitrowski [17], Nasr and Aouf [11], Aouf [10] and $\mathcal{C}^\lambda(1-\alpha) =: \mathcal{C}^\lambda(\alpha)$ $(0 \leq \alpha < 1)$ is known the $\lambda$-Robertson type functions class of order $\alpha$ and was studied by Chichra [5] and $\mathcal{C}^0(1-\alpha) =: \mathcal{C}(\alpha)$ $(0 \leq \alpha < 1)$ is called the starlike functions class of order $\alpha$ $(0 \leq \alpha < 1)$ and was studied by Robertson [13].

- $\mathcal{R}^0(b) =: \mathcal{R}^\lambda(b)$ is the close-to-convex functions class of complex order and was studied by Abdul Halim [7] and $\mathcal{R}^0(1-\alpha) =: \mathcal{R}(\alpha)$ $(0 \leq \alpha < 1)$ is said to be close-to-convex functions class of order $\alpha$ $(0 \leq \alpha < 1)$ (see, [16]).

- $\mathcal{D}^0(1-\alpha) =: \mathfrak{B}(\alpha)$ $(0 \leq \alpha < 1)$ is an analytic functions class and was studied by Chen [5].

In view of information between these definitions above and the main results, by selecting suitable values of the parameters in the main results, i.e., Theorems 2-4, several useful results can be easily revealed. It is impossible to list all of them, but we want only to present a number of the related results in the classes $\mathcal{R}^\lambda(b)$, $\mathcal{D}^\lambda(b)$, $\mathcal{S}^\lambda(b)$, $\mathcal{C}^\lambda(b)$, $\mathcal{R}(\alpha)$, $\mathcal{D}^\lambda(b)$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}(b)$ and also certain subclasses of these classes.

Firstly, by taking $w := 1$ in Theorems 1-3, respectively, the following corollaries are obtained.

**Corollary 1.** *Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $b \in \mathbb{C}^*$ and $\lambda \in \mathbb{R}$ with $|\lambda| < \pi/2$. If $f(z)$ satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)}\right)\right| \begin{cases} < \cos\lambda\left|\Im m\left(be^{-i\lambda}\right)\right| & if \;\; \Im m\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \;\; \Im m\left(be^{-i\lambda}\right) = 0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)}\right)\right| \begin{cases} < \cos\lambda\left|\Re e\left(be^{-i\lambda}\right)\right| & if \;\; \Re e\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \;\; \Re e\left(be^{-i\lambda}\right) = 0 \end{cases},$$

*then* $f(z) \in \mathcal{R}^{\lambda}(b)$.

**Corollary 2.** *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $b \in \mathbb{C}^*$ and $\lambda \in \mathbb{R}$ *with* $|\lambda| < \pi/2$. *If* $f(z)$ *satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{f'(z)-\frac{f(z)}{z}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)}\right)\right| \begin{cases} < \cos\lambda \left|\Im m\left(be^{-i\lambda}\right)\right| & if \ \ \Im m\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \ \ \Im m\left(be^{-i\lambda}\right) = 0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{f'(z)-\frac{f(z)}{z}}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{f(z)}{z}-1\right)}\right)\right| \begin{cases} < \cos\lambda \left|\Re e\left(be^{-i\lambda}\right)\right| & if \ \ \Re e\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \ \ \Re e\left(be^{-i\lambda}\right) = 0 \end{cases},$$

then $f(z) \in \mathcal{D}^{\lambda}(b)$.

**Corollary 3.** *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $b \in \mathbb{C}^*$, $\lambda \in \mathbb{R}$ $(|\lambda| < \pi/2)$ and also let $\mathcal{F}(z)$ be defined by

$$\mathcal{F}(z) = (1-\mu)f(z) + \mu z f'(z) \quad (0 \leq \mu \leq 1).$$

*If* $\mathcal{F}(z)$ *satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)}-\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)}\right)\right| \begin{cases} < \cos\lambda \left|\Im m\left(be^{-i\lambda}\right)\right| & if \ \ \Im m\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \ \ \Im m\left(be^{-i\lambda}\right) = 0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)}-\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)}\right)\right| \begin{cases} < \cos\lambda \left|\Re e\left(be^{-i\lambda}\right)\right| & if \ \ \Re e\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \ \ \Re e\left(be^{-i\lambda}\right) = 0 \end{cases},$$

*then* $\mathcal{F}(z) \in \mathcal{S}^{\lambda}(b)$.

Secondly, by letting $\mu := 0$ in Corollary 3, the following corollary is also received.

**Corollary 4.** *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $b \in \mathbb{C}^*$ and $\lambda \in \mathbb{R}$ with $|\lambda| < \pi/2$. *If the function* $f(z)$ *satisfies any one of the cases of the inequalities:*

$$\left|\Re e\left(\frac{\frac{z[zf'(z)]'}{f(z)}-\left(\frac{zf'(z)}{f(z)}\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)}\right)\right| \begin{cases} < \cos\lambda \left|\Im m\left(be^{-i\lambda}\right)\right| & if \ \ \Im m\left(be^{-i\lambda}\right) \neq 0 \\ \neq 0 & if \ \ \Im m\left(be^{-i\lambda}\right) = 0 \end{cases}$$

*and*

$$\left|\Im m\left(\frac{\frac{z[zf'(z)]'}{f(z)}-\left(\frac{zf'(z)}{f(z)}\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)}\right)\right|\begin{cases} <\cos\lambda\left|\Re e\left(be^{-i\lambda}\right)\right| & if\ \ \Re e\left(be^{-i\lambda}\right)\neq 0\\ \neq 0 & if\ \ \Re e\left(be^{-i\lambda}\right)=0\end{cases},$$

*then* $f(z)\in\mathcal{S}^{\lambda}(b)$.

Thirdly, by putting $\mu := 1$ in Corollary 3, the following corollary is then obtained.

**Corollary 5.** *Let* $f(z)\in\mathcal{A}$, $z\in\mathbb{U}$, $b\in\mathbb{C}^*$ and $\lambda\in\mathbb{R}$ with $|\lambda|<\pi/2$ . *If the function* $f(z)$ *satisfies any one of the cases of the following inequalities:*

$$\left|\Re e\left(\frac{\frac{z(zf'(z))''+(zf'(z))'}{f'(z)}-\left(\frac{zf''(z)}{f'(z)}+1\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf''(z)}{f'(z)}\right)}\right)\right|\begin{cases} <\cos\lambda\left|\Im m\left(be^{-i\lambda}\right)\right| & if\ \ \Im m\left(be^{-i\lambda}\right)\neq 0\\ \neq 0 & if\ \ \Im m\left(be^{-i\lambda}\right)=0\end{cases}$$

*and*

$$\left|\Im m\left(\frac{\frac{z(zf'(z))''+(zf'(z))'}{f'(z)}-\left(\frac{zf''(z)}{f'(z)}+1\right)^2}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf''(z)}{f'(z)}\right)}\right)\right|\begin{cases} <\cos\lambda\left|\Re e\left(be^{-i\lambda}\right)\right| & if\ \ \Re e\left(be^{-i\lambda}\right)\neq 0\\ \neq 0 & if\ \ \Re e\left(be^{-i\lambda}\right)=0\end{cases},$$

*then* $f(z)\in\mathcal{C}^{\lambda}(b)$.

Lastly, by setting $w := b$ in Theorems 1-3, respectively, the following corollaries are also obtained.

**Corollary 6.** Let $f(z)\in\mathcal{A}$, $z\in\mathbb{U}$ and $b\in\mathbb{C}^*$. *If the function* $f(z)$ *satisfies any one of the following inequalities:*

$$\left|\Re e\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)}\right)\right|<\frac{1}{2}\sin(2\lambda)\quad\left(|\lambda|<\pi/2;\ \lambda\neq 0\right)$$

*and*

$$\left|\Im m\left(\frac{zf''(z)}{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)}\right)\right|<\cos^2\lambda\quad\left(|\lambda|<\pi/2\right)$$

*then*

$$\Re e\left[\left(1+\frac{e^{i\lambda}}{b\cos\lambda}\left(f'(z)-1\right)\right)^b\right]>0\quad\left(z\in\mathbb{U}\right),$$

*where the values of the exponential form above is considered to be as its principal value.*

**Corollary 7.** Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ and $b \in \mathbb{C}^*$. *If the function $f(z)$ satisfies any one of the following inequalities:*

$$\left| \Re e \left( \frac{f'(z) - \frac{f(z)}{z}}{1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{f(z)}{z} - 1 \right)} \right) \right| < \frac{1}{2} \sin(2\lambda) \quad \Big( |\lambda| < \pi/2; \ \lambda \neq 0 \Big)$$

*and*

$$\left| \Im m \left( \frac{f'(z) - \frac{f(z)}{z}}{1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{f(z)}{z} - 1 \right)} \right) \right| < \cos^2 \lambda \quad \Big( |\lambda| < \pi/2 \Big)$$

*then*

$$\Re e \left[ \left( 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{f(z)}{z} - 1 \right) \right)^b \right] > 0 \quad \Big( z \in \mathbb{U} \Big),$$

*where the value of the exponential term above is chosen to be as its principal value.*

**Corollary 8.** *Let $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $b \in \mathbb{C}^*$ and let $\mathcal{F}(z)$, defined by $\mathcal{F}(z) = (1 - \mu) f(z) + \mu z f'(z)$ $(0 \leq \mu \leq 1)$, be satisfy any one of the following inequalities:*

$$\left| \Re e \left( \frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)} - \left( \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \right)^2}{1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1 \right)} \right) \right| < \frac{1}{2} \sin(2\lambda) \quad \Big( |\lambda| < \pi/2; \ \lambda \neq 0 \Big)$$

*and*

$$\left| \Im m \left( \frac{\frac{z[z\mathcal{F}'(z)]'}{\mathcal{F}(z)} - \left( \frac{zF'(z)}{\mathcal{F}(z)} \right)^2}{1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1 \right)} \right) \right| < \cos^2 \lambda \quad \Big( |\lambda| < \pi/2 \Big).$$

*Then,*

$$\Re e \left[ \left( 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1 \right) \right)^b \right] > 0 \quad \Big( z \in \mathbb{U} \Big),$$

*where each one of the values of the powers above is taken to be as its principal value.*

**Remark.** For the special cases of certain parameters of all theorems, for example, by taking $b - 1 = \lambda = 0$ in Theorems 1-3, Theorems 2-4 in [8] are received. In addition, by letting $\alpha = 0$ in all corollaries of the related theorems, some of the Corollaries 5-8 in [8] are also obtained.

## References

[1] F. M. Al-Oboudi and M. M. Haidan, *Spirallike functions of complex order*, J. Natural Geom. 19 (2000), 53-72.

[2] M. K. Aouf, F. M. Al-Oboudi, and M. M. Haidan, *On some results for $\lambda$-Roberston functions of complex order*, Publications De L'instutue Math. 75 (91) 2005, 93-98.

[3] M. K. Aouf, *p-valent classes related to convex functions of complex order*, Rocky Mountain J. Math. 16 (1985), 775-790.

[4] M. P. Chen, *On functions satisfying* $\Re e\{f(z)/z\} > \alpha$, Tamkang J. Math. 5 (1974), 231-234.

[5] P. N. Chichra, *Regular functions $f(z)$ for which $zf'(z)$ is $\alpha$-spirallike*, Proc. Amer. Math. Soc. 49 (1975), 151-160.

[6] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Heidelberg, Tokyo, 1983.

[7] S. Abdul Halim, *On a class of functions of complex order*, Tamkang J. Math. 30 (2) 1999, 147-153.

[8] H. Irmak and N. Tuneski, *Some Applications of the result of Nunokawa to certain normalized analytic functions*, Int. J. Math. Anal. (Ruse) 1 (7) 2007, 325-330.

[9] R. L. Libera, *Univalent $\alpha$-spiral functions*, Canad. J. Math. 19 (1967), 449-456.

[10] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, J. Natural Sci. Math. 25 (1985), 1-12.

[11] M. A. Nasr and M. K. Aouf, *On convex functions of complex order*, Mansoura Sci. Bull. Egypt 9 (1982), 565-582.

[12] M. Nunokawa, *On properties of non-Caratheodory functions*, Proc. Japan Acad. Ser. A Math. Sci., 68 (1992), 152–153.

[13] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. 37 (1936), no. 2, 374-408.

[14] M. S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. 16 (1969), 97-101.

[15] L. Spacek, *Contribution a la theori des function univalentes*, Casopis Pest. Math. (1932), 12-19.

[16] H. M. Srivastava and S. Owa, *Certain classes of analytic functions with varying arguments,* J. Math. Anal. Appl. 136 (1) (1988) 217-228.

[17] P. Waitrowski, *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Univ. Lodzk. Nauk. Math. Przyrod. Ser. II (39) (1971), 75-85.

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