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ON CERTAIN CLASSES OF P-VALENT FUNCTIONS INVOLVING DZIOK-SRIVASTAVA OPERATOR

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Abstract. This paper gives some inclusion relationships of certain class of $p-$valent functions which are defined by using the Dziok-Srivastava operator. Further, a property preserving integrals is considered. Some of our results generalize previously known results.

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1. Introduction

Let $A\left(p\right)$ denote the class of all functions of the form:

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\quad\left(p\in\mathbb{N}=\left\{1,2,3,...\right\};z\in U\right),\tag{1}$$

which are analytic and $p-$valent in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$ and let $A\left(1\right)=A$.

For function f given by (1) and g given by

$$g(z)=z^{p}+\sum_{n=1}^{\infty}b_{n+p}z^{n+p},\tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

$$\left(f*g\right)(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}=\left(g*f\right)(z).\tag{3}$$

For complex parameters $a_1, ..., a_q$ and $b_1, ..., b_s$($bj \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, ..., s$), the generalized hypergeometric function ${}_qF_s$ is defined by the following infinite series:

$$ {}_qF_s\left(a_1, ..., a_q; b_1, ..., b_s; z\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n .... \left(a_q\right)_n}{\left(b_1\right)_n ... \left(b_s\right)_n} \frac{z^n}{n!} \tag{4} $$

$$ \left(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U\right), $$

where $(\theta)_n$ is the Pochhammer symbol (or shift factorial) defined, in terms of the Gamma function Γ, by

$$ (\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0), \\ \theta\left(\theta+1\right)...\left(\theta+n-1\right) & (n \in \mathbb{N}). \end{cases} $$

Corresponding a function $h_p\left(a_1, ..., a_q; b_1, ..., b_s; z\right)$ defined by

$$ h_p\left(a_1, ..., a_q; b_1, ..., b_s; z\right) = z^p \; {}_qF_s\left(a_1, ..., a_q; b_1, ..., b_s; z\right) \quad (z \in U), \tag{5} $$

Dziok and Srivastava [5] (see also [6]) considered a linear operator

$$ H_p(a_1, ..., a_q; b_1, ..., b_s) : A\left(p\right) \rightarrow A\left(p\right) $$

defined by the following Hadamard product:

$$ H_p(a_1, ..., a_q; b_1, ..., b_s) f\left(z\right) = h_p\left(a_1, ..., a_q; b_1, ..., b_s; z\right) * f\left(z\right), \tag{6} $$

$$ \left(q \leq s+1; q, s \in \mathbb{N}_0; z \in U\right). $$

If $f \in A\left(p\right)$ is given by (1), then we have

$$ H_p(a_1, ..., a_q; b_1, ..., b_s) f(z) = z^p + \sum_{n=1}^{\infty} \Gamma_n\left[a_1; b_1\right] a_{n+p} z^{n+p} \left(z \in U\right), \tag{7} $$

where

$$ \Gamma_n\left[a_1; b_1\right] = \frac{\left(a_1\right)_n .... \left(a_q\right)_n}{\left(b_1\right)_n ... \left(b_s\right)_n} \frac{1}{n!} \quad \left(n \in \mathbb{N}\right). \tag{8} $$

To make the notation simple, we write

$$ H_{p,q,s}\left[a_1; b_1\right] f(z) = H_p(a_1, ..., a_q; b_1, ..., b_s) f(z). \tag{9} $$

It easily follows from (7) that

$$ z\left(H_{p,q,s}\left[a_1; b_1\right] f(z) \right)' = a_1 H_{p,q,s}\left[a_1+1; b_1\right] f(z) - (a_1 - p) H_{p,q,s}\left[a_1; b_1\right] f(z). \tag{10} $$

The linear operator $H_{p,q,s}[a_1;b_1]$ is a generalization of many other linear operators considered earlier.

Remark 1

(i) $H_{1,2,1}(a,b;c)f(z) = \left(I_c^{a,b}f\right)(z)\left(a,b\in\mathbb{C};c\notin\mathbb{Z}_0^-\right)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [9];

(ii) $H_{p,2,1}(n+p,1;1)f(z) = D^{n+p-1}f(z)(n>-p,p\in\mathbb{N})$, where the linear operator D^{n+p-1} was studied by Goel and Sohi [8]. In the case when $p=1$, $D^n f(z)$ is the Ruscheweyh derivative of $f(z)$ (see [14]);

(iii) $H_{p,2,1}(c+p,1;c+p+1)f(z) = \mathcal{F}_{c,p}(z) = \frac{c+p}{z^{\,c}}\int_0^z t^{\,c-1}f(t)dt\ (c>-p)$, where the operator $\mathcal{F}_{c,p}$ is the generalized Bernardi–Libera–Livingston integral operator (see [4]) and $\mathcal{F}_{c,1}=\mathcal{F}_c$ was introduced by Bernardi [1];

(iv) $H_{p,2,1}(p+1,1;p+1-\lambda)f(z) = \Omega_z^{(\lambda,p)}f(z)\ \ (0\le\lambda<1)$, where the operator $\Omega_z^{(\lambda,p)}$ was investigated by Srivastava and Aouf [16];

(v) $H_{p,2,1}(a,1;c)f(z) = L_p(a,c)f(z)(a\in\mathbb{R};c\in\mathbb{R}\backslash\mathbb{Z}_0^-)$, where the linear operator $L_p(a,c)$ was studied by Saitoh [15] which yields the operator $L(a,c)f(z)$ introduced by Carlson and Shaffer [2] for $p=1$;

(vi) $H_{1,2,1}(\mu,1;\lambda+1)f(z) = I_{\lambda,\mu}f(z)(\lambda>-1,\mu>0)$, where $I_{\lambda,\mu}$ is the Choi–Saigo–Srivastava operator [4] which is closely related to the Carlson–Shaffer [2] operator $L(\mu,\lambda+1)f(z)$;

(vii) $H_{p,2,1}(p+1,1;n+p)f(z) = I_{n+p-1}f(z)(n>-p;n\in\mathbb{Z})$, where I_{n+p-1} is the Noor operator of order $n+p-1$ which considered by Liu and Noor [12];

(viii) $H_{p,2,1}(\lambda+p,c;a)f(z) = I_p^\lambda(a,c)f(z)(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-;\lambda>-p)$, where $I_p^\lambda(a,c)$ is the Cho–Kwon–Srivastava operator [3].

Definition 1. We say that a function $f(z)\in A(p)$ is in the class $T_{p,q,s}^{\alpha}[a_1;b_1]$, if it satisfies the following condition:

$$\operatorname{Re}\left\{\frac{\left(\ H_{p,q,s}[a_1;b_1]f(z)\ \right)'}{pz^{p-1}}\right\}>\alpha \qquad (0\le\alpha<1;p\in\mathbb{N};z\in U). \tag{11}$$

Using (10), condition (11) can be re-written in the form

$$\operatorname{Re}\left\{a_1\frac{H_{p,q,s}[a_1+1;b_1]f(z)}{pz^p}-(a_1-p)\frac{H_{p,q,s}[a_1;b_1]f(z)}{pz^p}\right\}>\alpha \tag{12}$$
$$(0\le\alpha<1;p\in\mathbb{N};z\in U).$$

We note that:

$$T_{p,2,1}^{\alpha}(n+p,1;1) = T_{n+p-1}(\alpha) \qquad (n>-p,p\in\mathbb{N}),$$

where the class $T_{n+p-1}(\alpha)$ studied by Goel and Sohi [8].

2. Basic properties of the class $T^{\alpha}_{p,q,s}[a_1;b_1]$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $q \le s+1, q, s \in \mathbb{N}_0, 0 \le \alpha < 1$, $p \in \mathbb{N}$ and $a_1 > 0$.

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our inclusion theorems below.

Lemma 1 [10]. Let the (nonconstant) function $w(z)$ be analytic in U, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w^{'}(z_0) = \xi w(z_0)$, where ξ is a real number and $\xi \ge 1$.

Theorem 1. The following inclusion property holds true for the class $T^{\alpha}_{p,q,s}[a_1;b_1]$:

$$T^{\alpha}_{p,q,s}[a_1+1;b_1] \subset T^{\alpha}_{p,q,s}[a_1;b_1]. \tag{13}$$

Proof. Let $f(z) \in T^{\alpha}_{p,q,s}[a_1+1;b_1]$, and define a regular function $w(z)$ in U such htat $w(0) = 0, w(z) \neq -1$ by

$$a_1 H_{p,q,s}[a_1+1;b_1] f(z) - (a_1-p) H_{p,q,s}[a_1;b_1] f(z) = pz^p \frac{1+(2\alpha-1)w(z)}{1+w(z)}. \tag{14}$$

Differentiating (14) with respect to z, we obtain

$$\frac{(H_{p,q,s}[a_1+1;b_1] f(z))^{'}}{pz^{p-1}} = \frac{1+(2\alpha-1)w(z)}{1+w(z)} - \frac{2(1-\alpha)}{a_1} \frac{zw^{'}(z)}{(1+w(z))^2}. \tag{15}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have

$$z_0 w^{'}(z_0) = \xi w(z_0) \qquad (\xi \ge 1). \tag{16}$$

From (15) and (16), we have

$$\frac{(H_{p,q,s}[a_1+1;b_1] f(z_0))^{'}}{pz_0^{p-1}} = \frac{1+(2\alpha-1)w(z_0)}{1+w(z_0)} - \frac{2(1-\alpha)}{a_1} \frac{\xi w(z_0)}{(1+w(z_0))^2}. \tag{17}$$

Since $\mathrm{Re}\left\{\frac{1+(2\alpha-1)w(z_0)}{1+w(z_0)}\right\} = \alpha, \xi \ge 1$, and $\frac{\xi w(z_0)}{(1+w(z_0))^2}$ is real and positive, we see that $\mathrm{Re}\left\{\frac{(H_{p,q,s}[a_1+1;b_1] f(z_0))^{'}}{pz_0^{p-1}}\right\} < \alpha$, which obviously contradicts $f(z) \in T^{\alpha}_{p,q,s}[a_1+1;b_1]$. Hence $|w(z)| < 1$ for $z \in U$, and it follows from (14) that $f(z) \in T^{\alpha}_{p,q,s}[a_1;b_1]$. This completes the proof of Theorem 1.

Remark 2.

(i) Taking $q = 2$, $s = 1, a_1 = n + p$ $(n > -p)$ and $a_2 = b_1 = 1$ in Theorem 1 we obtain the result obtained by Goel and Sohi [8, Theorem 1];

(ii) Taking $q = 2$, $s = 1$ and $a_1 = a_2 = b_1 = 1$ in Theorem 1, then

$$T^{\alpha}_{p,2,1}(1,1;1) = T_p(\alpha) = \left\{ f(z) \in A(p) : \operatorname{Re}\left\{ \frac{f'(z)}{pz^{p-1}} \right\} > \alpha, \qquad 0 \le \alpha < 1 \right\}$$

and from Umezawa [17] such that functions are $p-$valent. Hence the $p-$valence of functions in the class $T_{p,q,s}[a_1; b_1]$ follows from (13).

Theorem 2. If $f(z) \in T^{\alpha}_{p,q,s}[a_1; b_1]$, then

$$\mathcal{F}_{c,p}(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \in T^{\alpha}_{p,q,s}[a_1; b_1], \text{ for } c > -p. \tag{18}$$

Proof. From (18), we have

$$z\left(H_{p,q,s}[a_1; b_1] \mathcal{F}_{c,p}(z) \right)' = (c+p) H_{p,q,s}[a_1; b_1] f(z) - c H_{p,q,s}[a_1; b_1] \mathcal{F}_{c,p}(z), \tag{19}$$

Define a regular function $w(z)$ in U such htat $w(0) = 0, w(z) \neq -1$ by

$$\frac{\left(H_{p,q,s}[a_1; b_1] \mathcal{F}_{c,p}(z) \right)'}{pz^{p-1}} = \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)}. \tag{20}$$

From (19) and (20) we have

$$(c+p) H_{p,q,s}[a_1; b_1] f(z) - c H_{p,q,s}[a_1; b_1] \mathcal{F}_{c,p}(z) = pz^p \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)}. \tag{21}$$

Differentiating (21) with respect to z, and using (20) we obtain

$$\frac{\left(H_{p,q,s}[a_1; b_1] f(z)\right)'}{pz^{p-1}} = \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)} - \frac{2(1-\alpha)}{c+p} \frac{zw'(z)}{(1 + w(z))^2}. \tag{22}$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1.

Remark 3. Taking $q = 2$, $s = 1, a_1 = n + p$ $(n > -p)$ and $a_2 = b_1 = 1$ in Theorem 2 we obtain the result obtained by Goel and Sohi [8, Theorem 2].

Theorem 3. If $f(z) \in A(p)$ and satisfy the condition

$$\operatorname{Re}\left\{ \frac{\left(H_{p,q,s}[a_1; b_1] f(z) \right)'}{pz^{p-1}} \right\} > \alpha - \frac{(1-\alpha)}{2(p+c)} \qquad (c > -p). \tag{23}$$

Then the function

$$\mathcal{F}_{c,p}(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \in T^{\alpha}_{p,q,s}[a_1; b_1].$$

The proof of Theorem 3 is similar to that of Theorem 2 and so we omit it.

Remark 4.

(i) Taking $q = 2,\ s = 1, a_1 = n + p\ (n > -p)$ and $a_2 = b_1 = 1$ in Theorem 3 we obtain the result obtained by Goel and Sohi [8, Theorem 3];

(ii) Taking $q = 2,\ s = 1, a_1 = a_2 = b_1 = 1,\ \alpha = 0$ and $c = 1$ in Theorem 3 we obtain the result obtained by Goel and Sohi [8, Corollary 3(a)];

(iii) Taking $q = 2,\ s = 1, a_1 = a_2 = b_1 = 1,\ \alpha = \frac{1}{2p+3}$ and $c = 1$ in Theorem 3 we obtain the result obtained by Goel and Sohi [8, Corollary 3(b)];

(iv) Taking $p = 1$ in (ii) and (iii), we get the extensions of an ealier result due to Libera [11] viz; $\mathrm{Re}\left\{f'(z)\right\} > 0$ implies $\mathrm{Re}\left\{(\mathcal{F}_c(z))'\right\} > 0$.

Theorem 4. Let $f(z)$ be defined by

$$\mathcal{F}_{c,p}(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \qquad (c > -p). \tag{24}$$

If $\mathcal{F}_{c,p}(z) \in T^{\alpha}_{p,q,s}[a_1; b_1]$, then $f(z) \in T^{\alpha}_{p,q,s}[a_1; b_1]$ in $|z| < \frac{c+p}{1+\sqrt{(c+p)^2+1}}$.

Proof. Since $\mathcal{F}_{c,p}(z) \in T_{p,q,s}[a_1; b_1]$ we can write

$$z\left(H_{p,q,s}[a_1; b_1]\, \mathcal{F}_{c,p}(z) \right)' = pz^p \left[\alpha + (1-\alpha)\, u(z)\right] \tag{25}$$

where $u(z) \in P$, the class of functions with positive real part in U and normalized by $u(0) = 1$. We can re-write (25) as

$$a_1 H_{p,q,s}[a_1+1; b_1]\, \mathcal{F}_{c,p}(z) - (a_1 - p) H_{p,q,s}[a_1; b_1]\, \mathcal{F}_{c,p}(z) = pz^p \left[\alpha + (1-\alpha)\, u(z)\right]. \tag{26}$$

Differentiating (26) with respect to z, and using (19) we obtain

$$\left(\frac{\left(H_{p,q,s}[a_1; b_1]\, f(z) \right)'}{pz^{p-1}} - \alpha \right) (1-\alpha)^{-1} = u(z) + \frac{1}{c+p} z u'(z). \tag{27}$$

Using the well-known estimate (see [13]) $\left| z u'(z) \right| \le \frac{2r}{1-r^2} \mathrm{Re}\, u(z)\,, |z| = r$, (27) yields

$$\mathrm{Re}\left\{ \left(\frac{\left(H_{p,q,s}[a_1; b_1]\, f(z) \right)'}{pz^{p-1}} - \alpha \right) (1-\alpha)^{-1} \right\} \ge \left(1 - \frac{1}{c+p} \frac{2r}{1-r^2} \right) \mathrm{Re}\, u(z). \tag{28}$$

The right-hand side of (28) is positive if $r < \frac{c+p}{1+\sqrt{(c+p)^2+1}}$. The result is sharp for the function $f(z)$ defined by

$$f(z) = \frac{1}{(c+p)\, z^{c-1}} \left(z^c \mathcal{F}_{c,p}(z)\right)'$$

where $\mathcal{F}_{c,p}(z)$ is given by $\left(H_{p,q,s}\,[a_1; b_1]\, \mathcal{F}_{c,p}(z) \right)' = pz^{p-1} \frac{1+(2\alpha-1)z}{1+z}$.

Remark 5.

(i) Taking $q = 2,\ s = 1, a_1 = n + p\ (n > -p)$ and $a_2 = b_1 = 1$ in Theorem 4 we obtain the result obtained by Goel and Sohi [8, Theorem 4];

(ii) Taking $q = 2,\ s = 1, a_1 = a_2 = b_1 = 1$ and $\alpha = 0$ in Theorem 4 we obtain the result obtained by Goel and Sohi [8, Corollary 4(a)];

(iii) By taking $c = 1$ in (ii), we obtain the result obtained by Goel [7].

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