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THIRD-ORDER DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS

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Abstract. Certain applications of third-order differential subordination results are obtained for normalized analytic functions in the open unit disk. These results are obtained by investigating appropriate classes of admissible functions. Several interesting examples are also discussed.

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1. Introduction, Preliminaries and Definitions

Let $\mathcal{A}$ denote the class of all normalized analytic functions $f(z)$ in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U}$. For $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{ f : f \in \mathcal{H}(\mathbb{U}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\},$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$.

Let f and F be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be *subordinate* to F, or (equivalently) F is said to be *superordinate* to f, if there exists a Schwarz function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write $f \prec F$ or $f(z) \prec F(z)$ $(z \in \mathbb{U})$. If the function F is univalent in $\mathbb{U}$, then we have

$$f \prec F \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

The Schwarzian derivative $\{f, z\}$ of an analytic, locally univalent function f is defined by

$$\{f, z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1}$$

and let the higher order Schwarzian derivatives are defined inductively (see [5])as:

$$\sigma_{n+1}(f(z)) = (\sigma_n(f(z)))' - (n-1)\frac{f''(z)}{f'(z)}\sigma_n(f(z)), \quad n \geq 3.$$

From (1), one can easily find

$$\sigma_3(f(z)) := \{f, z\},$$

$$\sigma_4(f(z)) = \frac{f''''(z)}{f'(z)} - 6\frac{f''(z)f'''(z)}{(f'(z))^2} + 6\left(\frac{f''(z)}{f'(z)}\right)^3.$$

Definition 1. *Let $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following (third-order) differential subordination:*

$$\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z) \quad (z \in \mathbb{U}), \tag{2}$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all $p(z)$ satisfying (2). A dominant $\tilde{q}(z)$ that satisfies

$$\tilde{q}(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all dominants $q(z)$ of (2) is said to be the best dominant.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \left\{\xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty\right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 2. *[1] Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$, and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition*

$$\psi(r, s, t, u; z) \notin \Omega,$$

whenever $r = q(\xi), \quad s = n\xi q'(\xi),$

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq n\,\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\} \quad \textit{and} \quad \mathrm{Re}\left\{\frac{u}{s}\right\} \geq n^2\,\mathrm{Re}\left\{\frac{\xi^2 q''(\xi)}{q'(\xi)}\right\}$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q)).$$

Lemma 1. *[1] Let $p \in \mathcal{H}[a, n]$ with $n \geq 2$, and let $q \in \mathcal{Q}(a)$ and satisfy*

$$\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0 \quad \textit{and} \quad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n,$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

The theory of differential subordination in $\mathbb{C}$ is a generalization of differential inequality in $\mathbb{R}$, and this theory of differential subordination was initiated by the works of Miller, Mocanu, and Reade in 1981. The monograph by Miller and Mocanu [3] gives a good introduction to the theory of differential subordination.

In recent years, various authors have successfully applied the theory of First and Second order differential subordination to address many important problem in this field. Recently, using second order differential subordination, Ali et. al. [2] obtained sufficient conditions involving the Schwarzian derivatives. For related work of Schwarzian derivative refer [3, 5].

Very recently, Antonino and Miller [1] obtained some general results of third-order differential inequalities and subordination for a large class. It should be remarked in passing that only few articles [1, 4] dealing with very narrow classes of third order differential subordination.

In this present investigation, by making use of the third-order differential subordination results of Antonino and Miller [1], we consider certain suitable classes of admissible functions and investigate some applications of third-order differential subordination of analytic functions. Several interesting examples are also discussed.

2. Subordination results

We first define the following class of admissible functions that are required in our first result.

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1$. The class of admissible functions $\Phi_S[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(v, w, x, y; z) \notin \Omega$$

whenever

$$v = q(\xi), \quad w = \frac{n\xi q'(\xi)}{q(\xi)} + q(\xi) \quad (q(\xi) \neq 0),$$

$$\operatorname{Re}\left\{\frac{2x + v^2 - 1 + 3(w-v)^2}{2(w-v)}\right\} \geq n \operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

and

$$\operatorname{Re}\left\{\frac{\begin{array}{c} v\{x[4(3w-2v)-6] + 2[2-3v-9v^2-8v^3] - 9w^2[5v+ \\ 1]\} + y + 2w[24v^3+13v^2+v-6] + 6w^3[3v-1] + 6w^2 \end{array}}{(w-v)}\right\} \geq n^2 \operatorname{Re}\left\{\frac{\xi^2 q''(\xi)}{q'(\xi)}\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ n \geq 2).$$

Theorem 2. *Let $\phi \in \Phi_S[\Omega, q]$. If $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ with*

$$\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ n \geq 2),$$

satisfies

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{3}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. Define the function p in $\mathbb{U}$ by

$$p(z) := \frac{zf'(z)}{f(z)}. \tag{4}$$

A simple calculation yields

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}. \tag{5}$$

Further computations show that

$$z^2\{f, z\} = \frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - \frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{1-(p(z))^2}{2}, \tag{6}$$

and

$$\begin{aligned} z^3\sigma_4(f(z)) &= z^3\left(\frac{f''''(z)}{f'(z)} - 6\frac{f''(z)f'''(z)}{(f'(z))^2} + 6\left(\frac{f''(z)}{f'(z)}\right)^3\right) \\ &= \frac{z^3p'''(z)}{p(z)} + 3\frac{z^2p''(z)}{p(z)} - 6\frac{z^3p'(z)p''(z)}{(p(z))^2} - 6\left(\frac{zp'(z)}{p(z)}\right)^2 + 6\left(\frac{zp'(z)}{p(z)}\right)^3 \\ &\quad + 3\frac{(zp'(z))^2}{p(z)} - 2z^2p''(z) - 2zp'(z) + (p(z))^3 - p(z). \end{aligned} \tag{7}$$

We now define the transformations from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned} &v = r, \quad w = r + \frac{s}{r}, \quad x = \frac{t+s}{r} - \frac{3}{2}\left(\frac{s}{r}\right)^2 + \frac{1-r^2}{2}, \quad \text{and} \\ &y = \frac{u+3(t+s^2)}{r} - \frac{6s(t+s)}{r^2} + 6\frac{s^3}{r^3} + r^3 - (2t+2s+r). \end{aligned} \tag{8}$$

Let
$\psi(r, s, t, u; z) = \phi(v, w, x, y; z)$

$$\begin{aligned} = \phi\Bigg(r, r + \frac{s}{r}, \frac{t+s}{r} - \frac{3}{2}\left(\frac{s}{r}\right)^2 + \frac{1-r^2}{2}, \frac{u+3(t+s^2)}{r} - \frac{6s(t+s)}{r^2} + 6\frac{s^3}{r^3} + r^3 \\ -(2t+2s+r); z\Bigg) \end{aligned} \tag{9}$$

The proof will make use of Lemma 1. Using (4), (5), (6), and (7), from (9) we obtain

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}, z^3\sigma_4(f(z)); z\right). \tag{10}$$

Hence from (3) and (10), we have

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

A computation using (8) yields

$$\frac{t}{s}+1 = \frac{2x+v^2-1+3(w-v)^2}{2(w-v)},$$

and

$$\frac{u}{s} = \frac{\begin{array}{c} v\{x[4(3w-2v)-6]+2[2-3v-9v^2-8v^3]-9w^2[5v+1]\} \\ +y+2w[24v^3+13v^2+v-6]+6w^3[3v-1]+6w^2 \end{array}}{(w-v)}$$

Thus the admissibility condition for $\phi \in \Phi_S[\Omega, q]$ in Definition 3 is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi_n[\Omega, q]$ and by Lemma 1

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}),$$

which evidently completes the proof of Theorem 2.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_S[h(\mathbb{U}), q]$ is written as $\Phi_S[h, q]$. The following result is an immediate consequence of Theorem 2.

Theorem 3. *Let $\phi \in \Phi_S[h, q]$. If $f \in \mathcal{A}$, $q \in \mathcal{Q}_1$ with*

$$\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ n \geq 2),$$

and

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}, z^3\sigma_4(f(z)); z\right)$$

is analytic in $\mathbb{U}$, *then*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}, z^3\sigma_4(f(z)); z\right) \prec h(z)$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Following similar arguments as in [1, Corollary 2.1, page 450], Theorem 3 can be extended to the following corollary where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 4. *Let* q *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$, *and for* $\rho \in (0,1)$ *set* $q_\rho(z) \equiv q(\rho z)$. *Let* $\phi \in \Phi_S[h, q_\rho]$. *If* $f \in \mathcal{A}$, $q_\rho \in \mathcal{Q}_1$ *with*

$$\operatorname{Re}\left\{\frac{\xi q''_\rho(\xi)}{q'_\rho(\xi)}\right\} \geq 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq n|q'_\rho(\xi)|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q_\rho);\ n \geq 2),$$

and

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}, z^3\sigma_4(f(z)); z\right)$$

is analytic in $\mathbb{U}$, *then*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}, z^3\sigma_4(f(z)); z\right) \prec h(z)$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Our next theorem yields the relation between the best dominant of the differential subordination and the solution of a corresponding differential equation.

Theorem 5. *Let* $\phi \in \Phi_S[h, q_\rho]$, $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ *and* $\phi\left(q(z), Q_1(z), Q_2(z), Q_3(z); z\right)$ *is analytic in* $\mathbb{U}$, *where*

$$Q_1(z) = q(z) + \frac{zq'(z)}{q(z)}$$

$$Q_2(z) = \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \frac{3}{2}\left(\frac{zq'(z)}{q(z)}\right)^2 + \frac{1-(q(z))^2}{2}$$

$$Q_3(z) = \frac{z^3q'''(z)}{q(z)} + 3\frac{z^2q''(z)}{q(z)} - 6\frac{z^3q'(z)q''(z)}{(q(z))^2} - 6\left(\frac{zq'(z)}{q(z)}\right)^2 + 6\left(\frac{zq'(z)}{q(z)}\right)^3 + 3\frac{(zq'(z))^2}{q(z)} - 2z^2q''(z) - 2zq'(z) + (q(z))^3 - q(z).$$

Let h *be univalent in* $\mathbb{U}$ *and suppose the differential equation*

$$\phi\left(q(z), Q_1(z), Q_2(z), Q_3(z); z\right) = h(z) \tag{11}$$

has a solution $q \in \mathcal{Q}_1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0 \quad \text{and} \quad \left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ n \geq 2),$$

then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) \prec h(z) \tag{12}$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

and $q(z)$ *is the best dominant.*

Proof. Following the same arguments as in [1, Theorem 3, page 451], By applying Theorem 2, we deduce that q is a dominant of (12). Since q also satisfies (11), it is also a solution of the differential subordination (12) and therefore q will be dominated by all dominants of (12). Hence q is the best dominant of (11).

We will apply Theorem 2 to a specific case for $q(z) = 1 + Mz$, $M > 0$.

In the particular case $q(z) = 1 + Mz, M > 0$, and in view of Definition 3, the class of admissible functions $\Phi_S[\Omega, q]$, denoted by $\Phi_S[\Omega, M]$, is described below.

Definition 4. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_S[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\Bigg(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{nM}{M + e^{-i\theta}}, \frac{Le^{-i\theta} + nM}{M + e^{-i\theta}} - \frac{3}{2}\left(\frac{nM}{M + e^{-i\theta}}\right)^2 - \frac{1}{2}Me^{i\theta} \times (2 + Me^{i\theta}), \frac{Ne^{-i\theta} + 3Le^{-i\theta} + 3n^2M^2e^{i\theta}}{M + e^{-i\theta}} - \frac{6nM}{(M + e^{-i\theta})^2}(Le^{-i\theta} + nM) + 6\left(\frac{nM}{M + e^{-i\theta}}\right)^3 - (2L + (2n+1)Me^{i\theta} + 1) + (1 + Me^{i\theta})^3; z\Bigg) \notin \Omega, \tag{13}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$, $\mathrm{Re}\,\{Le^{-i\theta}\} \geq n(n-1)M$, and $\mathrm{Re}\,\{Ne^{-i\theta}\} \geq 0$ for all θ, and $n \geq 2$.

Corollary 6. *Let $\phi \in \Phi_S[\Omega, M]$. If $f \in \mathcal{A}$ with*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq nM \quad (z \in \mathbb{U};\ n \geq 2;\ M > 0),$$

satisfies

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) \in \Omega,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

For the special case $\Omega = q(\mathbb{U}) = \{w : |w - 1| < M\}$, the class $\Phi_S[\Omega, M]$ is simply denoted by $\Phi_S[M]$.

Corollary 7. *Let $\phi \in \Phi_S[M]$. If $f \in \mathcal{A}$ with*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq nM \quad (z \in \mathbb{U};\ n \geq 2;\ M > 0),$$

satisfies

$$\left|\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) - 1\right| < M,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

Example 1. *The function* $\phi(v, w, x, y; z) = (1-\alpha)\dfrac{w}{v} + \alpha x$, $(0 \leq \alpha \leq 1)$ *satisfies the admissibility condition* (13) *and hence Corollary 7 yields*

$$\left|(1-\alpha)\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} + \alpha z^2\{f, z\} - 1\right| < M \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (M > 0).$$

When $\alpha = 1$, *we have*

$$\left|z^2\{f, z\} - 1\right| < M \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (M > 0).$$

When $\alpha = 0$, *we have*

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1\right| < M \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (M > 0).$$

This result was obtained in [2, Example 2.7, page 8].

Example 2. *The functions* $\phi_1(v, w, x, y; z) = \alpha v + \beta w + vx, (\alpha, \beta \in \mathbb{R})$ *and* $\phi_2(v, w, x, y; z) = v(w + x)$ *satisfy the admissibility condition* (13) *and hence Corollary 7 yields*

$$\left|\alpha\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{zf'(z)}{f(z)}z^2\{f, z\} - 1\right| < M \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M,$$

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\}\right) - 1\right| < M \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (M > 0).$$

We, next apply Theorem 2 to two important particular cases corresponding to $q(\mathbb{U})$ being a disk and $q(\mathbb{U})$ being a half-plane, considered in the next section.

3. Two Special Cases

Case 1. We specialize the class of admissible functions and corresponding theorems for the case of $q(\mathbb{U})$ being a disk. The function

$$q(z) = M\frac{Mz + a}{M + \overline{a}z} \quad (z \in \mathbb{U}; M > 0), \tag{14}$$

with $|a| < M$, is univalent in $\overline{\mathbb{U}}$ and satisfies $q(\mathbb{U}) = \mathbb{U}_M = \{w : |w| < M\}$, $q(0) = a, q \in \mathcal{Q}(a)$ and $E(q) = \emptyset$. Antonino and Miller [1] have proved the following lemma for this specific function q.

Lemma 8. *[1] Let q be given by* (14) *and $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ be analytic in $\mathbb{U}$, with $p(z) \not\equiv a$ and $n \geq 2$. If there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}_M$ and $w_0 \in \partial\mathbb{U}$ such that $p(z_0) = q(w_0), p(\mathbb{U}_{r_0}) \subset \mathbb{U}_M$, and*

$$|zp'(z)||M + \overline{a}e^{i\theta}|^2 \leq Mn\left[M^2 - |a|^2\right]$$

when $z \in \overline{\mathbb{U}}_{r_0}$ and $\theta \in [0, 2\pi]$, then

$$z_0 p'(z_0) = nq(w_0)\frac{|q(w_0) - a|^2}{|q(w_0)|^2 - |a|^2},$$

$$\operatorname{Re}\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n\frac{|q(w_0) - a|^2}{|q(w_0)|^2 - |a|^2}, \quad \textit{and}$$

$$\operatorname{Re}\frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq 6n^2 \operatorname{Re}\frac{[\overline{a}q(w_0) - |a|^2]^2}{[|q(w_0)|^2 - |a|^2]^2}.$$

We will use Lemma 8 and Definition 3 to define the class of admissible functions for the specific function q defined in (14). We denote $\Phi_S[\Omega, q]$ by $\Phi_S[\Omega, M, a]$. Since $q(z) = Me^{i\theta}$, with $\theta \in [0, 2\pi]$, when $|z| = 1$, by using Lemma 8, the conditions of admissibility as given in Definition 3 change as given in the following definition.

Definition 5. *Let Ω be a set in $\mathbb{C}$, let q be given by* (14)*, and let $n \geq 2$. The class of admissible functions $\Phi_S[\Omega, M, a]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(v, w, x, y; z) \notin \Omega$$

whenever

$$v = Me^{i\theta}, \quad w = n\frac{|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2} + Me^{i\theta},$$

$$\operatorname{Re}\left\{\frac{2x + v^2 - 1 + 3(w - v)^2}{2(w - v)}\right\} \geq n\,\frac{|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2},$$

$$\operatorname{Re}\left\{\frac{v\{x[4(3w - 2v) - 6] + 2[2 - 3v - 9v^2 - 8v^3] - 9w^2[5v + 1]\} + y + 2w[24v^3 + 13v^2 + v - 6] + 6w^3[3v - 1] + 6w^2}{(w - v)}\right\}$$
$$\geq \frac{6n^2}{[M^2 - |a|^2]^2}\operatorname{Re}\,[\overline{a}Me^{i\theta} - |a|^2]^2 \quad (z \in \mathbb{U};\ \theta \in [0, 2\pi]). \tag{15}$$

If $a = 0$, then from (15) we see that $\Phi_S[\Omega, M, 0]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, with $n \geq 2$, that satisfy the admissibility condition

$$\phi(Me^{i\theta}, n + Me^{i\theta}, L, N; z) \notin \Omega,$$

whenever

$$\begin{aligned} &\operatorname{Re}\left\{2L + (Me^{i\theta})^2 - 1 + 3n^2\right\} \geq 2n^2, \\ &\operatorname{Re}\Big\{N + 5(Me^{i\theta})^4 + (Me^{i\theta})^3[12n-7] + (Me^{i\theta})^2[9n^2 + 20n + 4L - 16] \\ &\quad + Me^{i\theta}[18n^3 - 27n^2 + 2n(6L+1) - 6L - 8] - 6n[n^2 - n + 2]\Big\} \geq 0, \\ &\textit{and} \quad \theta \in [0, 2\pi]. \end{aligned}$$

With this definition of the admissible class, we obtain the following differential subordination result from Theorem 2.

Theorem 9. *Let q be given by (14) and let $f \in \mathcal{H}[a, n]$ satisfy*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right|\left|M + \overline{a}e^{i\theta}\right|^2 \leq nM[M^2 - |a|^2],$$
$$(z \in \mathbb{U};\ n \geq 2;\ \theta \in [0, 2\pi]).$$

If Ω be a set in $\mathbb{C}$ and $\phi \in \Phi_S[\Omega, M, a]$, then

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) : z \in \mathbb{U}\right\} \subset \Omega,$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

In the special case when $a = 0$, the above theorem reduces to the following corollary.

Corollary 10. *Let $q(z) = Mz$ and let $f \in \mathcal{H}_0$ with $n \geq 2$ satisfy*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq nM \quad (z \in \mathbb{U}).$$

If Ω be a set in $\mathbb{C}$ and $\phi \in \Phi_S[\Omega, M, 0]$, then

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) : z \in \mathbb{U}\right\} \subset \Omega,$$

implies

$$\frac{zf'(z)}{f(z)} \prec Mz \quad (z \in \mathbb{U}).$$

Case 2. Now, we specialize the class of admissible functions and corresponding theorems for the case of $q(\mathbb{U})$ being the half-plane $\Delta = \{w : \mathrm{Re} w > 0\}$. The function

$$q(z) = \frac{a + \overline{a}z}{1 - z} \quad (z \in \mathbb{U}), \tag{16}$$

where $\mathrm{Re}\, a > 0$, is univalent in $\overline{\mathbb{U}} \setminus \{1\}$ and satisfies $q(\mathbb{U}) = \Delta$, $q(0) = a$, and $q \in \mathcal{Q}$. Straightforward calculations lead to

$$zq'(z) = \frac{-|q(z) - a|^2}{2\mathrm{Re}\,(a - q(z))},$$

$$\mathrm{Re}\, \frac{zq''(z)}{q'(z)} + 1 = 0,$$

$$\mathrm{Re}\, \frac{z^2 q'''(z)}{q'(z)} = \frac{3|q(z) - a|^2}{2(\mathrm{Re}\,(a))^2}.$$

Using these results in [1, Lemma D, page 445], we obtain the following lemma.

Lemma 11. *Let q be given by (16) and $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ be analytic in $\mathbb{U}$, with $p(z) \not\equiv a$ and $n \geq 2$. If there exist points $z_0 \in \mathbb{U}$ and $w_0 \in \partial\mathbb{U} \setminus \{1\}$ such that $p(z_0) = q(w_0)$, $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$, where $r_0 = |z_0|$, and*

$$|zp'(z)||1 - w|^2 \leq 2n|\mathrm{Re}\, a|,$$

then,

$$z_0 p'(z_0) = -\frac{n}{2} \frac{(a - q(w_0))(\overline{a} + q(w_0))}{\mathrm{Re}\,(a)},$$

$$\mathrm{Re}\, \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq 0,$$

$$\mathrm{Re}\, \frac{z^2 p'''(z_0)}{p'(z_0)} \geq \frac{3n^2}{2} \frac{|a - q(w_0)|^2}{(\mathrm{Re}\,(a))^2}.$$

We will use Lemma 11 and Definition 3 to define the class of admissible functions for the specific function q defined in (16). We denote $\Phi_S[\Omega, q]$ by $\Phi_S[\Omega, a]$ and when $\Omega = \Delta$, denote the class by $\Phi_S[a]$.

Definition 6. *Let Ω be a set in $\mathbb{C}$, let q be given by* (16)*, and let $n \geq 2$. The class of admissible functions $\Phi_S[\Omega, a]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(v, w, x, y; z) \notin \Omega$$

whenever

$$v = i\rho, \quad w = n\frac{i|a - i\rho|^2}{2\rho \mathrm{Re}\, a} + i\rho,$$

$$\mathrm{Re}\left\{\frac{2x + v^2 - 1 + 3(w-v)^2}{2(w-v)}\right\} \geq 0,$$

$$\mathrm{Re}\left\{\frac{v\{x[4(3w-2v)-6] + 2[2-3v-9v^2-8v^3] - 9w^2[5v+1]\} + y + 2w[24v^3+13v^2+v-6] + 6w^3[3v-1] + 6w^2}{(w-v)}\right\}$$

$$\geq \frac{3n^2}{2}\frac{|a-i\rho|^2}{(\mathrm{Re}\, a)^2}, \quad (z \in \mathbb{U};\ \rho \in \mathbb{R};\ a \in \mathbb{C}).$$

In this particular case, Theorem 2 can be rephrased in the following form.

Theorem 12. *Let q be given by* (16) *and let $f \in \mathcal{A}$ satisfy*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| |1-\xi|^2 \leq 2n|\mathrm{Re}\, a|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus \{1\};\ n \geq 2).$$

(i) If Ω be a set in $\mathbb{C}$ and $\phi \in \Phi_S[\Omega, a]$, then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) \in \Omega,$$

implies $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{U})$.

(ii) If $\phi \in \Phi_S[a]$, then

$$\mathrm{Re}\ \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}, z^3\sigma_4(f(z)); z\right) > 0,$$

implies $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{U})$.

Example 3. *Let* $\alpha : \mathbb{U} \to \mathbb{R}$ *and let* $\operatorname{Re} a > 0$. *If* $\phi(v,w,x,y;z) = v + \alpha(z)\frac{w}{v}$, $\Omega = \{w : \operatorname{Re} w < 1\}$. *To use part(i) of Theorem 12 we need to show that* $\phi \in \Phi_S[\Omega, a]$, *that is, the admissibility condition* (6) *is satisfied. This follows since*

$$\begin{aligned}\operatorname{Re}\ [\phi(v,w,x,y;z)] &= \operatorname{Re}\ \left[v + \alpha(z)\frac{w}{v}\right] \\ &= \operatorname{Re}\ \left[i\rho + \alpha(z)\frac{n(a^2+\rho^2)}{2\rho^2 \operatorname{Re} a} + 1\right] \geq 1.\end{aligned}$$

Hence from part(i) of Theorem 12 we obtain:

Let $f \in \mathcal{A}$ *satisfy*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| |1-\xi|^2 \leq 2n|\operatorname{Re} a|,$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus \{1\};\ n \geq 2).$$

If $\phi \in \Phi_S[\Omega, a]$, *then*

$$\frac{zf'(z)}{f(z)} + \alpha(z)\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \prec \frac{2z}{z-1} \quad \Rightarrow \quad \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}.$$

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