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SOME SUBORDINATIONS RESULTS FOR CERTAIN SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper we derive several subordination results for certain classes of analytic functions of complex order.

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1. Introduction

Let A denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$. We also denote by K the class of function $f(z) \in A$ that are convex in U.

Let $P(\lambda, b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy :

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right\} > 0$$

$$(z \in U; b \in C^* = C\backslash\{0\}; 0 \leq \lambda \leq 1) \tag{1.2}$$

or which satisfy the following inequality :

$$\left| \frac{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - 1}{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - 1 + 2b} \right| < 1\,. \tag{1.3}$$

Also, a function $f(z) \in A$ is said to be in the class $R(\lambda, b)$ if it satisfies :

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(f'(z) + \lambda z f''(z) - 1\right)\right\} > 0$$

$$(z \in U; b \in C^*; 0 \leq \lambda \leq 1) \tag{1.4}$$

or which satisfy the following inequality :

$$\left| \frac{f'(z) + \lambda z f''(z) - 1}{f'(z) + \lambda z f''(z) - 1 + 2b} \right| < 1 \,. \tag{1.5}$$

We note that :

(i) $$P(0,b) = S(b) = \left\{ f \in A : \mathrm{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1 \right) \right] > 0, z \in U, b \in C^* \right\}, \tag{1.6}$$

where $S(b)$, is the class of starlike functions of complex order, studied by Nasr and Aouf [6] and Owa [7];

(ii) $$P(1,b) = C(b) = \left\{ f \in A : \mathrm{Re}\left(1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \right) > 0, z \in U, b \in C^* \right\}, \tag{1.7}$$

where $C(b)$, is the class of convex functions of complex order, studied by Nasr and Aouf [5] and Owa [7];

(iii) $$R(0,b) = R(b) = \left\{ f \in A : \mathrm{Re}\left[1 + \frac{1}{b}(f'(z) - 1) \right] > 0, z \in U, b \in C^* \right\}, \tag{1.8}$$

where $R(b)$ is the class of close-to-convex functions of complex order, studied by Halim [3] and Owa [7].

Definition 1. ***(Hadamard Product or Convolution).*** *Given two functions f and g in the class A, where $f(z)$ is given by (1.1) and $g(z)$ is given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \,. \tag{1.9}$$

The Hadamard product (or convolution) $(f * g)(z)$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in U) \,.$$

Definition 2. ***(Subordination Principal).*** *For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)\,(z \in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))\,(z \in U)$. Indeed it is known that $f(z) \prec g(z) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.*

Furthermore, if the function g is univalent in U, then we have the following equivalence [4, p. 4] :

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 3. ***(Subordinating Factor Sequence).*** *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ is of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in U; a_1 = 1). \tag{1.10}$$

Lemma 1. *[10]. The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \quad (z \in U).$$

In [1], Altintas and Qzkan studied the classes $P(\lambda, b)$ and $R(\lambda, b)$ when $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \, (a_n \geq 0)$ and obtained the following lemmas :

Lemma 2. *[1]. If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \, (a_n \geq 0) \in P(\lambda, b)$, then we have*

$$\sum_{n=2}^{\infty} [1 + \lambda(n-1)] \, (n + |b| - 1) a_n \leq \frac{|b|^2}{\operatorname{Re}(b)}.$$

Lemma 3. *[1]. If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \, (a_n \geq 0) \in R(\lambda, b)$, then we have*

$$\sum_{n=2}^{\infty} n \, [1 + \lambda(n-1)] \, a_n \leq \frac{|b|^2}{\operatorname{Re}(b)}.$$

In [8], Ozkan used Lemma 2 and Lemma 3 to obtain subordination results involving the Hadamard product of the above classes. All the results obtained by Ozkan [8, Theorem 2.1 and Theorem 2.8] are not correct because Lemma 1 and Lemma 2 are proved by Altinatas and Ozkan [1] when $f(z)$ has negative coefficients, i. e., $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \, (a_n \geq 0)$.

Now, we prove the following lemmas which give a sufficient conditions for functions belonging to the classes $P(\lambda, b)$ and $R(\lambda, b)$.

Lemma 4. *Let the function $f(z)$ which is defined by (1.1) satisfies the following condition :*

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)]\left[(n-1)+|2b+n-1|\right]|a_n| \leq 2\,|b| \quad (\lambda \geq 0; b \in C^*), \tag{1.11}$$

then $f(z) \in P(\lambda, b)$.

Proof. Suppose that the inequality (1.11) holds. Then we have for $z \in U$,

$$\begin{aligned}
&\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}-1\right|-\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}+2b-1\right| \\
=\;& \left|\left[zf'(z)+\lambda z^2 f''(z)\right]-\left[(1-\lambda)f(z)+\lambda z f'(z)\right]\right|- \\
& \left|\left[zf'(z)+\lambda z^2 f''(z)\right]+(2b-1)\left[(1-\lambda)f(z)+\lambda z f'(z)\right]\right| \\
=\;& \left|\sum_{n=2}^{\infty}(n-1)\left[1+\lambda(n-1)\right]a_n z^n\right|-\left|2bz+\sum_{n=2}^{\infty}\left[1+\lambda(n-1)\right](2b+n-1)a_n z^n\right| \\
\leq\;& |z|\left\{\sum_{n=2}^{\infty}(n-1)\left[1+\lambda(n-1)\right]|a_n|\,|z|^{n-1}-\right. \\
& \left\{2\,|b|-\sum_{n=2}^{\infty}\left[1+\lambda(n-1)\right]|2b+n-1|\,|a_n|\,|z|^{n-1}\right\} \\
& \qquad \leq \sum_{n=2}^{\infty}\left[1+\lambda(n-1)\right]\left[(n-1)+|2b+n-1|\right]\Bigg\}|a_n|-2\,|b| \leq 0\,,
\end{aligned}$$

which shows that $f(z)$ belongs to the class $P(\lambda, b)$.

Lemma 5. *Let the function $f(z)$ which is defined by (1.1) satisfies the following condition :*

$$\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]|a_n| \leq |b|\,, \tag{1.12}$$

then $f(z) \in R(\lambda, b)$.

Proof. Suppose that the inequality (1.12) holds. Then we have for $z \in U$,

$$\begin{aligned}
&\left|f'(z)+\lambda z f''(z)-1\right|-\left|f'(z)+\lambda z f''(z)+2b-1\right| \\
=\;& \left|\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]a_n z^{n-1}\right|-\left|2b+\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]a_n z^{n-1}\right|
\end{aligned}$$

$$\leq \sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]|a_n|\,|z|^{n-1} - \left\{2\,|b| - \sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]|a_n|\,|z|^{n-1}\right\}$$

$$\leq 2\left\{\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right]|a_n| - |b|\right\} \leq 0\,,$$

which shows that $f(z)$ belongs to the class $R(\lambda, b)$.

Let $P^*(\lambda, b)$ and $R^*(\lambda, b)$ denote the classes of functions $f(z) \in A$ whose coefficients satisfy the conditions (1.11) and (1.12), respectively. We note that $P^*(\lambda, b) \subseteq P(\lambda, b)$ and $R^*(\lambda, b) \subseteq R(\lambda, b)$.

2. Main Results

Employing the technique used earlier by Attiya [2] and Srivastava and Attiya [9], we prove:

Theorem 6. *Let $f(z) \in P^*(\lambda, b)$. Then, for the function $g \in K$*

$$\left(\frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}\right)(f*g)(z) \prec g(z) \quad (z \in U) \tag{2.1}$$

and

$$\mathrm{Re}(f(z)) > -\frac{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}{(\lambda+1)\left[1+|2b+1|\right]} \quad (z \in U)\,. \tag{2.2}$$

The constant factor $\frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}$ in the subordination result (2.1) cannot be replaced by a larger one.

Proof. Let $f(z) \in P^*(\lambda, n)$ and let $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in K$. Then we have

$$\begin{aligned} &\frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}(f*g)(z) \\ &= \frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right). \end{aligned} \tag{2.3}$$

Thus, by Definition 3, the subordination result (2.1) will hold true if the sequence

$$\left\{\frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b| + (\lambda+1)[1+|2b+1|]\right\}}a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality :

$$\operatorname{Re}\left\{1+\frac{(\lambda+1)\,[1+|2b+1|]}{2\,\{2|b|+(\lambda+1)[1+|2b+1|]\}}\sum_{n=1}^{\infty}a_n z^n\right\}>0 \quad (z\in U)\,. \tag{2.5}$$

Now, since

$$\Psi(n)=[1+\lambda(n-1)]\,[(n-1)+|2b+n-1|]$$

is an increasing function of $n\,(n\ge 2)$, we have

$$\begin{aligned}
&\operatorname{Re}\left\{1+\frac{(\lambda+1)\,[1+|2b+1|]}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}\sum_{n=1}^{\infty}a_n z^n\right\}\\
=\;&\operatorname{Re}\left\{1+\frac{(\lambda+1)\,[1+|2b+1|]}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}z+\right.\\
&\left.\frac{1}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}\sum_{n=2}^{\infty}(\lambda+1)\,[1+|2b+1|]\,a_n z^n\right\}\\
\ge\;&1-\frac{(\lambda+1)\,[1+|2b+1|]}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}r\\
&-\frac{1}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}\sum_{n=2}^{\infty}[1+\lambda(n-1)][(n-1)+|2b+n-1||a_n|r^n\\
>\;&1-\frac{(\lambda+1)\,[1+|2b+1|]}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}r-\frac{2\,|b|}{\{2|b|+(\lambda+1)[1+|2b+1|]\}}r\\
=\;&1-r>0 \quad (|z|=r<1)\,,
\end{aligned}$$

where we have also made use of assertion (1.11) of Lemma 4. Thus (2.5) holds true in U. This proves the inequality (2.1). The inequality (2.2) follows from (2.1) by taking the convex function $g(z)=\dfrac{z}{1-z}=z+\sum\limits_{n=2}^{\infty}z^n$. To prove the sharpness of the constant $\dfrac{(\lambda+1)\,[1+|2b+1|]}{2\,\{2|b|+(\lambda+1)[1+|2b+1|]\}}$, we consider the function $f_0(z)\in P^*(\lambda,b)$ given by

$$f_0(z)=z-\frac{2\,|b|}{(\lambda+1)[1+|2b+1|]}z^2\,. \tag{2.6}$$

Thus from (2.1), we have

$$\frac{(\lambda+1)\,[1+|2b+1|]}{2\,\{2|b|+(\lambda+1)[1+|2b+1|]\}}f_0(z)\prec\frac{z}{1-z} \quad (z\in U)\,. \tag{2.7}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.6) that

$$\min_{|z|\le r}\left\{\operatorname{Re}\frac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b|+(\lambda+1)[1+|2b+1|]\right\}}f_0(z)\right\}=-\frac{1}{2}\,. \tag{2.8}$$

This shows that the constant $\dfrac{(\lambda+1)\left[1+|2b+1|\right]}{2\left\{2|b|+(\lambda+1)[1+|2b+1|]\right\}}$ is the best possible.

Putting $\lambda=0$ in Theorem 1, we obtain the following result.

Corollary 7. *Let the function $f(z)$ defined by (1.1) be in the class $P^*(0,b)=S^*(b)$ and suppose that $g(z)\in K$. Then*

$$\left(\frac{[1+|2b+1|]}{2\left[2|b|+1+|2b+1|\right]}\right)(f*g)(z)\prec g(z)\quad (z\in U) \tag{2.9}$$

and

$$\operatorname{Re}(f(z))>-\frac{[2|b|+1+|2b+1|]}{[1+|2b+1|]}\quad (z\in U)\,.$$

The constant factor $\dfrac{[1+|2b+1|]}{2\left[2|b|+1+|2b+1|\right]}$ in the subordination result (2.9) cannot be replaced by a larger one.

Putting $\lambda=1$ in Theorem 1, we obtain the following result.

Corollary 8. *Let the function $f(z)$ defined by (1.1) be in the class $P^*(1,b)=C^*(b)$ and suppose that $g(z)\in K$. Then*

$$\left(\frac{1+|2b+1|}{2\left[|b|+1+|2b+1|\right]}\right)(f*g)(z)\prec g(z)\quad (z\in U) \tag{2.10}$$

and

$$\operatorname{Re}(f(z))>-\frac{|b|+1+|2b+1|}{1+|2b+1|}\quad (z\in U)\,.$$

The constant factor $\dfrac{1+|2b+1|}{2\left[|b|+1+|2b+1|\right]}$ in the subordination result (2.10) cannot be replaced by a larger one.

Remark 1. *Putting (i) $\lambda=0$ and $b=1-\alpha, 0\le\alpha<1$ (ii) $\lambda=1$ and $b=1-\alpha, 0\le\alpha<1$ (iii) $\lambda=0$ and $b=1$ (iv) $\lambda=b=1$ in Theorem 1, we obtain the results obtained by Ozkan [8, Corollaries 2.4, 2.5, 2.6 and 2.7, respectively] .*

Theorem 9. *Let $f(z) \in R^*(\lambda, b)$. Then, for the function $g \in K$*

$$\left(\frac{(1+\lambda)}{[2(1+\lambda)+|b|]}\right)(f*g)(z) \prec g(z) \quad (z \in U) \tag{2.11}$$

and

$$\mathrm{Re}(f(z)) > -\frac{[1(1+\lambda)+|b|]}{2(1+\lambda)} \quad (z \in U)\,. \tag{2.12}$$

The constant factor $\frac{(1+\lambda)}{[2(1+\lambda)+|b|]}$ *in the subordination result (2.11) cannot be replaced by a larger one.*

Proof. Let $f(z) \in R^*(\lambda, b)$ and let $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in K$. Then we have

$$\frac{(1+\lambda)}{[2(1+\lambda)+|b|]}(f*g)(z) = \frac{(1+\lambda)}{[2(1+\lambda)+|b|]}\left(z + \sum_{n=2}^{\infty} a_n c_n z^k\right). \tag{2.13}$$

Thus, by Definition 3, the subordination result (2.11) will hold if the sequence

$$\left\{\frac{(1+\lambda)}{[2(1+\lambda)+|b|]}a_n\right\}_{n=1}^{\infty} \tag{2.14}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality :

$$\mathrm{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{2(1+\lambda)}{[2(1+\lambda)+|b|]} a_n z^n\right\} > 0 \quad (z \in U)\,. \tag{2.15}$$

Now, since

$$\Phi(n) = n\,[1+\lambda(n-1)]$$

is an increasing function of $n\,(n \geq 2)$, we have

$$\begin{aligned}
& \mathrm{Re}\left\{1 + \frac{(1+\lambda)}{[2(1+\lambda)+|b|]}\sum_{n=1}^{\infty} a_n z^n\right\} \\
= \;& \mathrm{Re}\left\{1 + \frac{2(1+\lambda)}{[2(1+\lambda)+|b|]}z + \frac{1}{[2(1+\lambda)+|b|]}\sum_{n=2}^{\infty} 2(1+\lambda) a_n z^n\right\} \\
\geq \;& 1 - \frac{2(1+\lambda)}{[2(1+\lambda)+|b|]}r - \frac{1}{[2(1+\lambda)+|b|]}\sum_{n=2}^{\infty} n\,[1+\lambda(n-1)]\,|a_n|\,r^n \\
> \;& 1 - \frac{2(1+\lambda)}{[2(1+\lambda)+|b|]}r - \frac{|b|}{[2(1+\lambda)+|b|]}r \\
= \;& 1 - r > 0 \quad (|z| = r < 1)\,,
\end{aligned}$$

where we have also made use of assertion (1.12) of Lemma 5. Thus (2.15) holds true in U. This proves the inequality (2.11). The inequality (2.12) follows from (2.11) by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum\limits_{n=2}^{\infty} z^n$. To prove the sharpness of the constant $\frac{(1+\lambda)}{2(1+\lambda)+|b|}$, we consider the function $f_1(z) \in R^*(\lambda, b)$ given by

$$f_1(z) = z - \frac{|b|}{2(1+\lambda)} z^2 . \tag{2.16}$$

Thus from (2.11), we have

$$\frac{(1+\lambda)}{[2(1+\lambda)+|b|]} f_1(z) \prec \frac{z}{1-z} \quad (z \in U) . \tag{2.17}$$

Moreover, it can easily be verified for the function $f_1(z)$ given by (2.16) that

$$\min_{|z| \le r} \left\{ \operatorname{Re} \frac{(1+\lambda)}{[2(1+\lambda)+|b|]} f_1(z) \right\} = -\frac{1}{2} . \tag{2.18}$$

This shows that the constant $\frac{(1+\lambda)}{[2(1+\lambda)+|b|]}$ is the best possible.

Putting $\lambda = 0$ in Theorem 2, we obtain the following result.

Corollary 10. *Let the function $f(z)$ defined by (1.1) be in the class $R^*(0, b) = R^*(b)$ and suppose that $g(z) \in K$. Then*

$$\left(\frac{1}{2+|b|} \right) (f * g)(z) \prec g(z) \quad (z \in U) \tag{2.19}$$

and

$$\operatorname{Re}(f(z)) > -\frac{2+|b|}{2} \quad (z \in U) . \tag{2.20}$$

The constant factor $\frac{1}{2+|b|}$ in the subordination result (2.19) cannot be replaced by a larger one.

Remark 2. *(i) Putting $b = 1-\alpha$, $0 \le \alpha < 1$ and (ii) $b = 1$ in Corollary 3, we obtain the results obtained by Ozkan [8, Corollary 2.10 and Corollary 2.11, respectively].*

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