SOME NEW GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper, we define the sequence spaces : $c_0(M_k, \Delta_u^n, p, q), c(M_k, \Delta_u^n, p, q)$ and $l_\infty(M_k, \Delta_u^n, p, q)$, where for any sequence $x = (x_n)$, the difference sequence Δx is given by $\Delta x = (\Delta x_n)_{n=1}^\infty = (x_n - x_{n-1})_{n=1}^\infty$. We also examine some inclusion relations between these spaces and discuss some properties and results related to them. These spaces will give as a special cases the spaces defined and studied by Tripathy and Sarma in 2005 and some others before.

2000 *Mathematics Subject Classification*:40A05, 40C05, 46A45.

Keywords: Difference sequence spaces, a sequence of Orlicz functions.

1. Introduction and definitions

Let w, c, c_0 and l_∞ denote the spaces of all, convergent, null and bounded sequences respectively. Throughout this article $p = (p_k)$ is a sequence of strictly positive real numbers and (p_k^{-1}) will be denoted by (t_k).

A paranorm on a linear topological space X is a function $g : X \to \mathbb{R}$ which satisfies the following axioms :

for any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$,

(i) $g(\theta) = 0$, where $\theta = (0, 0, 0, \cdots)$, the zero sequence,

(ii) $g(x) = g(-x)$,

(iii) $g(x + y) \leq g(x) + g(y)$ (subadditivity),

and

(iv) the scalar multiplication is continuous, that is,

$$\lambda \to \lambda_0, x \to x_0 \text{ imply } \lambda x \to \lambda_0 x_0 \ ;$$

in other words,

$$| \lambda - \lambda_0 | \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and is written (X, g).

Any function g which satisfies all the conditions (i)-(iv) together with the condition :

(v) $g(x) = 0$ if and only if $x = \theta$,

is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (see Maddox [4]).

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If convexity of M is replaced by $M(x+y) \leq M(x)+M(y)$, then it is called a modulus function, defined and studied by Nakano [6], Ruckle [7], Maddox [5] and others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of h, if there exist a constant $K > 0$ such that

$$M(2h) \leq KM(h) \ (h \geq 0).$$

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of the inequality

$$M(lh) \leq KlM(h),$$

for all values of h and for $l > 1$.

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct the Orlicz sequence space :

$$l_M := \{x = (x_k) : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

which is a Banach space with the norm :

$$\| x \|_M = \inf\{\rho \ > 0 : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) \leq 1\}.$$

If $M(x) = x^p, 1 \leq p < \infty$, the space l_M coincide with the classical sequence space l_p.

Let (X, q) be a seminormed space seminormed by q. Then Tripathy and Sarma [8] defined the sequence spaces $c_0(M, \Delta, p, q), c(M, \Delta, p, q)$ and $l_\infty(M, \Delta, p, q)$.

Now, let $M = (M_k)$ be a sequence of Orlicz functions, n is a nonnegative integer and $u = (u_k)$ is any sequence such that $u_k \neq 0$ for each k, then we define the following sequence spaces :

$$c_0(M_k, \Delta_u^n, p, q) = \{x = (x_k) : [M_k(q(\frac{\Delta_u^n x_k}{\rho}))]^{p_k} t_k \to 0, \text{ as } k \to \infty, \text{ for some } \rho > 0\},$$

$$c(M_k, \Delta_u^n, p, q) = \{x = (x_k) : [M_k(q(\frac{\Delta_u^n x_k - le}{\rho}))]^{p_k} t_k \to 0,$$
$$\text{as } k \to \infty, \text{ for some } \rho > 0 \text{ and some } l \in \mathbb{C} \},$$

and

$$l_\infty(M_k, \Delta_u^n, p, q) = \{x = (x_k) : \sup_k [M_k(q(\frac{\Delta_u^n x_k}{\rho}))]^{p_k} t_k < \infty, \text{ for some } \rho > 0\},$$

where $e = (1, 1, 1, \cdots)$ and

$$\begin{aligned} \Delta_u^0 x_k &= u_k x_k, \\ \Delta_u^1 x_k &= u_k x_k - u_{k+1} x_{k+1}, \\ \Delta_u^2 x_k &= \Delta(\Delta_u^1 x_k), \\ &\vdots \\ \Delta_u^n x_k &= \Delta(\Delta_u^{n-1} x_k), \end{aligned}$$

so that

$$\Delta_u^n x_k = \Delta_{u_k}^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} u_{k+r} x_{k+r}.$$

If $M_k = M$ for each k, $n = 0$ and $u = e$, then these gives the spaces of Tripathy and Sarma [8].

2. Main results

We need the following inequality (see Tripathy and Sarma [8])
Let $p = (p_k)$ be any sequence of strictly positive real numbers, $H = \sup_k p_k$ and $D = \max(1, 2^{H-1})$, then
$| a_k + b_k |^{p_k} \leq D[| a_k |^{p_k} + | b_k |^{p_k}]$.
Now, We prove the following theorems :

Theorem 1 *For any sequence $p = (p_k)$ of strictly positive real numbers, the sequence spaces $c_0(M_k, \Delta_u^n, p, q), c(M_k, \Delta_u^n, p, q)$ and $l_\infty(M_k, \Delta_u^n, p, q)$ are linear spaces over the set of complex numbers.*

Proof: We shall prove only for $c_0(M_k, \Delta_u^n, p, q)$. The others can be treated similarly. Let $x = (x_k), y = (y_k) \in c_0(M_k, \Delta_u^n, p, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists some positive ρ_1 and ρ_2 such that :
$[M_k(q(\frac{\Delta_u^n x_k}{\rho_1})]^{p_k} t_k \to 0$, as $k \to \infty$
and $[M_k(q(\frac{\Delta_u^n y_k}{\rho_2})]^{p_k} t_k \to 0$, as $k \to \infty$ Define $\rho = \max(2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2)$. Then we have

$$\begin{aligned}
& [M_k(q(\frac{\alpha\Delta_u^n x_k + \beta\Delta_u^n y_k}{\rho})]^{p_k} t_k \\
\leq \; & [M_k(q(\frac{\alpha\Delta_u^n x_k}{\rho}) + q(\frac{\beta\Delta_u^n y_k}{\rho}))]^{p_k} t_k \\
\leq \; & [M_k(q(\frac{\Delta_u^n x_k}{2\rho_1}) + q(\frac{\Delta_u^n y_k}{2\rho_2}))]^{p_k} t_k \\
\leq \; & \frac{1}{2^{p_k}}[M_k(q(\frac{\Delta_u^n x_k}{\rho_1}) + q(\frac{\Delta_u^n y_k}{\rho_2}))]^{p_k} t_k \\
\leq \; & D[M_k(q(\frac{\Delta_u^n x_k}{\rho_1}))]^{p_k} t_k + D[M_k(q(\frac{\Delta_u^n y_k}{\rho_2}))]^{p_k} t_k \\
\to \; & 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence $\alpha x + \beta y \in c_0(M_k, \Delta_u^n, p, q)$.

Theorem 2 *The space $l_\infty(M_k, \Delta_u^n, p, q)$ is a paranormed space with the paranorm*

$$g(x) = q(x_1) + \inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\},$$

where $j = \max(1, H)$, $H = \sup_k p_k$.

Proof:

$$\begin{aligned} g(\theta) &= q(0) + \inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\theta}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} \\ &= 0. \end{aligned}$$

$$\begin{aligned} g(-x) &= q(-x_1) + \inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n(-x_k)}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} \\ &= q(x_1) + \inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} \\ &= g(x). \end{aligned}$$

Let $x = (x_k)$, $y = (y_k) \in l_\infty(M_k, \Delta_u^n, p, q)$. Then there exists some $\rho_1 > 0$ and $\rho_2 > 0$ such that :
$M_k(q(\frac{\Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1$ and $M_k(q(\frac{\Delta_u^n y_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1$.
Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned} &\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n(x_k + \Delta_u^n y_k)}{\rho})t_k^{\frac{1}{p_k}}\} \\ \leq\ &\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho}) + q(\frac{\Delta_u^n y_k}{\rho}))t_k^{\frac{1}{p_k}}\} \\ \leq\ &\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho}))t_k^{\frac{1}{p_k}}\} + \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n y_k}{\rho}))t_k^{\frac{1}{p_k}}\} \\ \leq\ &\frac{\rho_1}{\rho_1+\rho_2}\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho}))t_k^{\frac{1}{p_k}}\} + \frac{\rho_2}{\rho_1+\rho_2}\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n y_k}{\rho}))t_k^{\frac{1}{p_k}}\} \\ \leq\ &\frac{\rho_1}{\rho_1+\rho_2} + \frac{\rho_2}{\rho_1+\rho_2} = 1. \end{aligned}$$

Now, we have

$$\begin{aligned} g(x+y) &= q(x_1+y_1) + \inf\{(\rho_1+\rho_2)^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k + \Delta_u^n y_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1\} \\ &\leq q(x_1) + \inf\{\rho_1^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho_1})t_k^{\frac{1}{p_k}}\} \leq 1\} \\ +q(y_1) + \inf\{\rho_2^{\frac{p_k}{j}} &: \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n y_k}{\rho_2})t_k^{\frac{1}{p_k}}\} \leq 1\} \\ &= g(x) + g(y). \end{aligned}$$

Finally, Let $\eta \in \mathbb{C}$. Then the continuity of the product follows from the following inequality :

$$\begin{aligned} g(\eta x) &= q(\eta x_1) + \inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\eta \Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} \\ &= \mid \eta \mid q(x_1) + \inf\{(\mid \eta \mid r)^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} \\ &= \mid \eta \mid g(x), \end{aligned}$$

where $\frac{1}{r} = \frac{|\eta|}{\rho}$.

Theorem 3 *Let $p = (p_k)$ be a bounded sequence. Then the sequence spaces $c_0(M_k, \Delta_u^n, p, q)$, $c(M_k, \Delta_u^n, p, q)$ and $l_\infty(M_k, \Delta_u^n, p, q)$ are complete paranormed spaces paranormed by g given in Theorem 2 .*

Proof: We prove it for the case $l_\infty(M_k, \Delta_u^n, p, q)$. The others are similar.
Let (x^i) be a Cauchy sequence in $l_\infty(M_k, \Delta_u^n, p, q)$, where $(x^i) = (x^i)_{k=1}^\infty$, for all $i \in \mathbb{N}$. Then $g(x^i - x^j) \to 0$ as $i, j \to \infty$.
For a given $\varepsilon > 0$, let r, u_0 and x_0 be fixed such that $\frac{\varepsilon}{ru_0x_0} > 0$ and $M_k(\frac{ru_0x_0}{2}) \geq \sup_{k\geq 1}(p_k)^{t_k}$.
Now $g(x^i - x^j) \to 0$ as $i, j \to \infty$ implies that there exists $m_0 \in \mathbb{N}$ such that $g(x^i - x^j) < \frac{\varepsilon}{ru_0x_0}$for all $i, j \geq m_0$.
Therefore we obtain that $g(x_1^i - x_1^j) < \frac{\varepsilon}{ru_0x_0}$ and

$$\inf\{\rho^{\frac{p_k}{j}} : \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1, \rho \geq 0\} < \frac{\varepsilon}{ru_0x_0}.$$

Since $g(x_1^i - x_1^j) < \frac{\varepsilon}{ru_0x_0}$for all $i, j \geq m_0$, we get that (x_1^i) is a Cauchy sequence in $\mathbb{C}$. This implies that (x_1^i) is convergent in $\mathbb{C}$.
Let $\lim_{i\to\infty} x_1^i = x_{\ 1}$, then we have $\lim_{j\to\infty} g(x_1^i - x_1^j) < \frac{\varepsilon}{ru_0x_0}$ which imply that $g(x_1^i - x_1) < \frac{\varepsilon}{ru_0x_0}$.
But $M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1$, then letting $\rho = g(x^i - x^j)$, we see that $M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{g(x^i - x^j)})t_k^{\frac{1}{p_k}}\} \leq 1$. This implies that $M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{g(x^i - x^j)})t_k^{\frac{1}{p_k}}\} \leq p_k^{t_k} \leq M_k(\frac{ru_0x_0}{2})$. Now $q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{g(x^i - x^j)}) \leq \frac{ru_0x_0}{2}$ yields that $q(\Delta_u^n x_k^i - \Delta_u^n x_k^j) \leq g(x^i - x^j)\frac{ru_0x_0}{2} < \frac{\varepsilon}{ru_0x_0}\frac{ru_0x_0}{2} = \frac{\varepsilon}{2}$. Therefore $(\Delta_u^n x_k^i)$ is a Cauchy sequence in $\mathbb{C}$ for all

$k \in \mathbb{N}$. This implies that $(\Delta_u^n x_k^i)$ is convergent in $\mathbb{C}$. Now let $\lim_{i\to\infty} \Delta_u^n x_k^i = \Delta_u^n x_k$, for all $k \in \mathbb{N}$. Then we have

$$\lim_{j\to\infty} \sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k^j}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1$$

which implies that

$$\sup_{k\geq 1}\{M_k(q(\frac{\Delta_u^n x_k^i - \Delta_u^n x_k}{\rho})t_k^{\frac{1}{p_k}}\} \leq 1.$$

Let $i \geq m_0$. Then taking infimum of such ρ's, we have $g(x^i - x) < \varepsilon$.

Hence $x = x^i - (x^i - x) \in l_\infty(M_k, \Delta_u^n, p, q)$ since $l_\infty(M_k, \Delta_u^n, p, q)$ is a linear space.

Therefore $l_\infty(M_k, \Delta_u^n, p, q)$ is complete.

Theorem 4 *Let* $0 < p_k \leq r_k$ *for all* $k \in \mathbb{N}$. *Then* $c_0(M_k, \Delta_u^n, p, q) \subseteq c_0(M_k, \Delta_u^n, r, q)$.

Proof: Let $x = (x_k) \in c_0(M_k, \Delta_u^n, p, q)$. Then there exists some $\rho > 0$ such that $\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k = 0$, and this implies that $[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k \leq 1$, for sufficiently large k since (M_k) is a sequence of nondecresing Orlicz functions. Therefore $\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{r_k} t_k \leq \lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k = 0$.

This proves that $x = (x_k) \in c_0(M_k, \Delta_u^n, r, q)$ and completes the proof.

Theorem 5 *(i) Let* $0 < \inf p_k \leq p_k \leq 1$. *Then* $c_0(M_k, \Delta_u^n, p, q) \subseteq c_0(M_k, \Delta_u^n, q)$.
(ii) Let $1 \leq p_k \leq \sup p_k < \infty$. *Then* $c_0(M_k, \Delta_u^n, q) \subseteq c_0(M_k, \Delta_u^n, p, q)$.

Proof: (i) Let $x = (x_k) \in c_0(M_k, \Delta_u^n, p, q)$. Then $\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k = 0$.
This gives that

$$\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]t_k \leq \lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k = 0.$$

Hence $x = (x_k) \in c_0(M_k, \Delta_u^n, q)$.

(ii) Let $p_k \geq 1$ for all k , $\sup_k p_k < \infty$ and let $x = (x_k) \in c_0(M_k, \Delta_u^n, q)$. Then for each $\varepsilon(0 < \varepsilon < 1)$ there exists a positive integer N such that

$$\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]t_k \leq \varepsilon < 1.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k}t_k \leq \lim_{k\to\infty}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]t_k \leq \varepsilon < 1.$$

Hence $x = (x_k) \in c_0(M_k, \Delta_u^n, p, q)$.

Theorem 6 *Let $n \geq 1$. Then for all $0 \leq i \leq n$, $Z(M_k, \Delta_u^i, p, q) \subseteq Z(M_k, \Delta_u^n, p, q)$, where $Z = l_\infty, c, c_0$.*

Proof: We show that $c_0(M_k, \Delta_u^{n-1}, p, q) \subseteq c_0(M_k, \Delta_u^n, p, q)$.
Let $x = (x_k) \in c_0(M_k, \Delta_u^{n-1}, p, q)$. Then we have
$[M_k(q(\frac{\Delta_u^{n-1}x_k}{\rho})]^{p_k}t_k \to 0$ as $k \to \infty$ for some $\rho > 0$.
Since (M_k) is a sequence of nondecreasing convex functions, we have

$$\begin{aligned}
[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k}t_k &= [M_k(q(\frac{\Delta_u^{n-1}x_k - \Delta_u^{n-1}x_{k+1}}{\rho})]^{p_k}t_k \\
&\leq [M_k(q(\frac{\Delta_u^{n-1}x_k + \Delta_u^{n-1}x_{k+1}}{\rho})]^{p_k}t_k \\
&\leq D[M_k(q(\frac{\Delta_u^{n-1}x_k}{\rho})]^{p_k}t_k + D[M_k(q(\frac{\Delta_u^{n-1}x_{k+1}}{\rho})]^{p_k}t_k \\
&\to 0 \text{ as } k \to \infty \text{ for some } \rho > 0.
\end{aligned}$$

Therefore $x = (x_k) \in c_0(M_k, \Delta_u^n, p, q)$.

Hence the result follows by mathematical induction.

Theorem 7 *Let $M = (M_k)$ be a sequence of Orlicz functions such that M_k satisfies the Δ_2-condition for each k. Then $c_0(M_k, \Delta_u^n, p, q) \subseteq c(M_k, \Delta_u^n, p, q) \subseteq l_\infty(M_k, \Delta_u^n, p, q)$.*

Proof: Let $x = (x_k) \in c_0(M_k, \Delta_u^n, p, q)$. Then $x = (x_k) \in c(M_k, \Delta_u^n, p, q)$.
Let $x = (x_k) \in c(M_k, \Delta_u^n, p, q)$. Then we have

$$\begin{aligned}
[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k}t_k &= [M_k(q(\frac{\Delta_u^n x_k - l + l}{\rho})]^{p_k}t_k, \text{ for some } l \in \mathbb{C} \\
&\leq D[M_k(q(\frac{\Delta_u^n x_k - l}{\rho})]^{p_k}t_k + D[M_k(q(\frac{l}{\rho})]^{p_k}t_k \\
&\leq D[M_k(q(\frac{\Delta_u^n x_k - l}{\rho})]^{p_k}t_k + D[\frac{l}{\rho}K\delta^{-1}M_k(2]^H t_k,
\end{aligned}$$

where $H = \sup_k p_k$, $D = \max(1, 2^{H-1})$.
Hence we get that $x = (x_k) \in l_\infty(M_k, \Delta_u^n, p, q)$.

Theorem 8 *Let* $M = (M_k)$ *be a sequence of Orlicz functions such that* M_k *satisfies the* Δ_2-*condition for each* k*. Then* $Z(\Delta_u^n, q) \subseteq Z(M_k, \Delta_u^n, p, q)$*, where* $Z = l_\infty, c$ *and* c_0.

Proof: We prove it for the case $l_\infty(\Delta_u^n, q) \subseteq l_\infty(M_k, \Delta_u^n, p, q)$.

Let $x = (x_k) \in l_\infty(\Delta_u^n, q)$. Then there exists $L > 0$ such that $q(\Delta_u^n x_k) \leq L$, for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned}[M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k &\leq [M_k(\frac{L}{\rho})]^{p_k} t_k \\ &\leq [KhM_k(L)], \text{ for all } k \in \mathbb{N}, \text{ using } \Delta_2 - \text{condition}.\end{aligned}$$

Hence $\sup_k [M_k(q(\frac{\Delta_u^n x_k}{\rho})]^{p_k} t_k < \infty$. Thus $l_\infty(\Delta_u^n, q) \subseteq l_\infty(M_k, \Delta_u^n, p, q)$.

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