SOME OBSERVATIONS ON BANACH LATTICES

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Abstract. In this note, our aim is to solve a problem in Banach lattices with topologically full centre which is posed by A.W.Wickstead.

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1. Introduction

An operator $T : E \to E$ on a real or a complex Riesz space is called central if it is dominated by a multiple of the identity operator. That is, T is central operator if and only if there exists some scalar $\lambda > 0$ such that $|Tx| \leq \lambda |x|$ holds for all $x \in E$.

The collection of all central operators is denoted by $Z(E)$ and is referred to as the center of the Banach lattice E.
Each cenral operator is regular and every positive central operator is a lattice homomorphism. The center $Z(E)$ is an ideal in $L_r(E)$,the vector space of all regular operators on E. The central operators are special examples of operators known as orthomorphism.
An order bounded operator $T : E \to E$ on a Riesz space is said to be an orthomorphism if $|x| \wedge |y| = 0$ implies $|x| \wedge |T_y| = 0$. Recall also that an operator $T : E \to E$ on a Riesz space is band preserving if $T(B) \subseteq B$ for every band B of E. It is worth pointing out that in general a band preserving operator does not need to be order bounded.However ,for Banach lattices the situation is better.Namely,the band preserving operators and the central operators coincide.
Abromovich-Veksler-Koldunov showed that[3] ,for a given operator $T : E \to E$ on a Banach lattice

1. T is central

2. T is band preserving

3. T is an ortomorphism

In spite of the fact that for an arbitrary Riesz space E, the partially ordered vector space $L_r(E)$ need not to be a Riesz space, the center $Z(E)$ is always a Riesz space.
Wickstead showed that the center $Z(E)$ equipped with the operator norm is an AM -space with unit. The unit is the identity operator I, so that for each operator $T \in Z(E)$ we have
$\|T\| = \||T|\| = inf\{\lambda \geq 0 : |T| \leq \lambda I\}$
$Z(E)$ is a unital Banach algebra.
$Z(E)_c$ will denote the commutant of $Z(E)$ in $L_b(E)$, the vector space of all order bounded operators on E.
That is $Z(E)_c = \{T \in L_b(E) : TS = ST, S \in Z(E)\}$
The center $Z(E)$ of a Banach lattice E is topologically full [1,definition 1.3] if whenever $0 \leq x \leq y$, $x, y \in E$, there is a sequence (T_n) in $Z(E)$ such that $T_n y \to x$.
If $0 \leq x \leq y$, $x, y \in E$ and $T_n y \to x$, then $(T_n^+ \wedge I)\, y = (T_n y)^+ \wedge y \to x \wedge y = x$, so
If $Z(E)$ is topologically full, we may assume that $0 \leq T_n \leq I$ for all $n \in N$. I will use the notation
$Z(E)_+$ for the set $\{T \in Z(E) : 0 \leq T \leq I\}$.

Proposition 1 *If E is a Dedekind complete Banach lattice, then $Z(E)$ is a maximal abelian subalgebra of $L(E)$.*

Proof: $Z(E)$ is an ideal of E. As a matter of fact, $Z(E) = E_I$, the principal ideal generated by I in $L_r(E)$
Let $x, y \in E$ and $T \in Z(E)_c$ we must prove that, $x \perp y \Rightarrow x \perp T_y$. Suppose, E is a Dedekind complete Banach lattice
$0 \leq T \in B(I) = Z(E) = Z(E)_c$
$T \wedge nI \uparrow T$
$x \wedge y = 0 \Rightarrow x \wedge (T \wedge nI)\, y = 0$, for all $n \in N$
$x \wedge y = 0 \Rightarrow x \wedge T_y = 0$

Proposition 2 *If E is a Dedekind σ- complete Banach lattice, then* $Z(E) = Z(E)_c$

Proof: $Z(E) \subset Z(E)_c$ its clearly known, so

let $T \in Z(E)_c$ and $x \perp y$

If P is a projection band onto the principal band generated by x, then $P \in Z(E)$

$Px = x$ and $Pz = 0 \Leftrightarrow x \perp y$

$P(Ty) = T(Py) = T(0) = 0$, then

$Ty \perp x$, T is band preserving operator therefore T is central.

Proposition 3 *If E is any Banach lattice has projection property such that $Z(E)$ topologically full, $x, y \in E$ with $0 \leq x \leq y$, then there is T in the commutant of $Z(E)$ such that $Ty = x$.*

Proof: Let $0 \leq x \leq y$, $T_n : E \to E$ and $P : E \to E$ band projection.

$P(E) = B$, $B \oplus B^d = Y \in E$,

now fix $y \in Y$,

$y = x + z$, $x \in B$, $z \in B^d$,

$(T_n P)\, y = T_n(P_y) = T_n(x)$, $n \to \infty$,

$Px = x$ so, $T_n(x) \to x$.

$|T_n(x) - P(y)| = |T_n(x) - x + x - P(y)| \leq |T_n(x) - x| + |P(y) - x|$,

$|T_n(x) - x| \to 0$, $|P(y) - x| \to 0$,iken $\|T_n(x) - P(y)\| \to 0$

Wickstead showed that[3], If E is a Banach lattice with topologically full center, then $Z(E)$ maximal abelian subalgebra of $L(E)$,

Proposition 4 *Let E is a Banach lattice, $x, y \in E$, $(T_n) \in Z(E)_+$*

$T_n(y) \to y$ and $T_n(x) \to 0$ then $x \perp y$.

Proposition 5 *If E is a Banach lattice $x, y \in E$, $(T_n) \in Z(E)_+$ and $T_n(y) \to y$, $T_n(x) \to 0$ then $Z(E)_c^+$ is a maximal abelian subalgebra of $L(E)$.*

Proof: Suppose $x, y \in E_+$ with $x \perp y$ and $S \in Z(E)_c^+$,

$T_n(Sx) = S(T_n x) \to S(0) = 0$, so $Sx \perp y$.
A positive element u in a Banach lattice is a topological order unit (=quasi-interior point of the positive cone) if the closed ideal generated by u is the whole space.i.e.$\overline{E_u} = E$. A Banach lattice with a topological order unit u may be represented as an order ideal in $C_\infty(K)$, the continuous extended real-valued functions on some compact Hausdorff space K which are finite on a dense subset of K, with u corresponding to the constantly one function on K, The centre may be identified with $C(K)$ and is topologically full.

Proposition 6 *If a positive element $u > 0$ is a topological order unit in a Banach lattice E, then for each $x \in E^+$ we have $||x \wedge nu - x|| \to 0$.*

Proof: Let $y \in E_+$ and fix $\epsilon > 0$, there exists some $y \in E_u$, such that $\|x - y\| < \epsilon$. From the lattice inequality $|x^+ \wedge y - y| \leq |x - y|$ by replacing x by $x^+ \wedge y$ that we can assume $0 \leq x \leq y$

Now fix some k such that $0 \leq x \leq ku$ and so $x \leq y \wedge ku$, hence if $n \geq k$, than the inequality

$0 \leq y - y \wedge nu \leq y - y \wedge ku \leq y - x$

$\|y - y \wedge nu\| < \epsilon$, therefore $\|y - y \wedge nu\| \to 0$.

Proposition 7 *If $u > 0$ is a topological order unit then,$u \vee y$ is topological order unit.i.e. $\overline{E_{u \vee y}} = E$.*

Proof: $\forall x \in E_+$
$\|y \wedge n(u \vee y) - y\| = \|y \wedge (nu) \vee y \wedge (ny) - y\| = \|(y \wedge nu - y) \vee (y \wedge ny - y)\|$

since $y - y \wedge nu \to 0$ and $y \wedge ny = y$ and by the order continuty of the lattice operations,

$\|y - y \wedge n(u \vee y)\| \to 0$ so, $\overline{E_{uvy}} = E$.

Proposition 8 *If A Banach lattice with topological order unit, then it has a topologically full center.*

Proof Represent E as an ideal in $C_\infty(K)$ with $u \vee y$ corresponding to 1_K

As $0 \leq x \leq y \leq 1_K$, both $x, y \in C(K)$ for all $n \in N$

let $T_n = \frac{x}{y+\frac{1}{n}1_K}$ so $T_n \in Z(E)$

Considering values pointwise,

$0 \leq T_n y - x = \left(\frac{x}{y+\frac{1}{n}1_K}\right) y - x = \frac{1}{n}\left(\frac{x}{y+\frac{1}{n}1_K}\right) \leq \frac{1}{n}1_K$ so,

$\|T_n y - x\| \leq \frac{1}{n}\|1_K\|$ and therefore, $T_n y \to x$

Proposition 9 *If E is a Banach lattice with topologically full centre and $x \in E$, there is a sequence in $Z(E)$ such that $T_n x \to |x|$.*

Proof: Since $Z(E)$ is topologically full, whenever $0 \leq x^+ \leq |x|$, there is a sequence (P_n) in $Z(E)$ with $P_n|x| \to x^+$.

For (P_n) is band preserving $P_n x^- \perp x^+$ so, $P_n x^- \perp P_n x^+$

$|P_n|x| - x^+| = |P_n(x^+ + x^-) - x^+| = |(P_n x^+ - x^+) + P_n x^-| = |P_n x^+ - x^+| + |P_n x^-|$

so, $P_n x^+ \to x^+$ and $P_n x^- \to 0$ so that $P_n x = P_n(x^+ - x^-) \to x^+$

and whenever $0 \leq x^- \leq |x|$ we can take (S_n) in $Z(E)$ with $S_n|x| \to x^-$ so that, $(P_n + S_n)x \to |x|$.

For all $n \in N$, we may assume $T_n = P_n + S_n$ so that $\exists T_n \in Z(E)$, $T_n x \to |x|$.

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