ON CERTAIN CLASSES OF UNIFORMLY ω-STARLIKE AND ω-CONVEX FUNCTIONS DEFINED BY CONVOLUTION

Abiodun Tinuoye Oladipo

Abstract. The author here wish to further study the new concept of analytic and univalent functions normalized with $f(\omega)$ and $f'(\omega)-1=0$ introduced by Kanas and Ronning in 1999 which was later also investigated in 2005 by Acu and Owa. The author in this paper obtain coefficient estimates, distortion theorems, convex linear combinations, and radii of ω-close-to-convex, $\omega - starlikeness$ and $\omega - convex$ for functions belonging to the subclass $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ of uniformly $\omega - starlikeness$ and $\omega - convex$ functions.We also consider integral operators associated with functions in this class.

2000 *Mathematics Subject Classification*: Primary 30C45.

1. Introduction

Let $A(\omega)$ denote the class of functions of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k \tag{1}$$

analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized with $f(\omega)$ and $f'(\omega)-1=0$. Also let $S(\omega) \subset A(\omega)$ denote the class of the univalent functions in $A(\omega)$. Kanas and Ronning [9] introduced, defined and studied the following classes.

$$S(\omega) = \{f \in \Gamma(\omega) : f \;\; is \;\; univalent \;\; in \;\; U\}$$

$$ST(\omega) = S^*(\omega) = \left\{ f(z) \in S(\omega) : Re\frac{(z-\omega)f'(z)}{f(z)} > 0, z \in U \right\}$$

$$CV(\omega) = S^c(\omega) = \left\{ f(z) \in S(\omega) : 1 + Re\frac{(z-\omega)f'(z)}{f'(z)} > 0, z \in U \right\}$$

These classes are respectively called ω-univalent, $\omega-$ *starlike* and $\omega-$ *convex* functions.
The class $S^*(\omega)$ is defined by geometric property that the image of any circular arc centered at ω is starlike with respect to $f(\omega)$ and the corresponding class $S^c(\omega)$ is defined by the property that the image of circular arc centred at ω is convex. We also observe that these definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A.W.Goodman [7] and [8], except that, in this case the point ω is arbitrarily fixed in U. Acu and Owa [1], Oladipo [13] and [14], Aouf et al [5] also did several works in this direction and they obtained many valuable results.
The aim of the author here is to continue the investigation of the univalent functions normalized with $f(\omega)=f'(\omega)-1=0$ where ω is an arbitrary fixed point in U.
Let $f\in S(\omega)$ be given by (1) and $g(z)\in S(\omega)$ be given by

$$g(z)=(z-\omega)+\sum_{k=2}^{\infty}b_k(z-\omega)^k \quad (b_k\geq 0) \tag{2}$$

then, the Hadamand product (or convolution) $f*g$ of f and g is defined as

$$(f*g)(z)=(z-\omega)+\sum_{k=2}^{\infty}a_kb_k(z-\omega)^k=(g*f)(z) \tag{3}$$

Following Goodman [7] and [8], Kanas and Ronning [9], Acu and Owa [1] and with the application of Ruschewey derivative operator [16], Oladipo [13] introduced and studied the following classes
(1)A function $f(z)$ of the form (1) is in the class $\Phi(\omega,\beta,b,m)$ if it satisfied the condition

$$Re\left\{1-\frac{2}{b}+\frac{2}{b}\frac{D_{\omega}^{m+1}f(z)}{D_{\omega}^{m}f(z)}\right\}>\beta$$

where $b\neq 0$, and $m>-1$, $0\leq\beta<1$, ω is a fixed point in U and $D_{\omega}^{m}f(z)$ is the ω-modified Ruscheweyh derivative.
(ii) A function $f(z)$ of the form (1) is said to be in the class $\Phi(\omega,\beta,\alpha,b,m)$ if and only if

$$Re\left\{1-\frac{2}{b}+\frac{2}{b}\frac{D_{\omega}^{m+1}f(z)}{D_{\omega}^{m}f(z)}\right\}>\alpha\left|\frac{2D_{\omega}^{m+1}f(z)}{D_{\omega}^{m}f(z)}-1\right|+\beta$$

where $b \neq 0$, $\alpha \geq 0$, $0 \leq \beta < 1$, $m > -1$, $D_\omega^m f(z)$ is the ω-modified of Ruscheweyh derivative operator. With special choices of the parameters involved he obtained several existing and new classes.
To further our investigation on the class of functions normalized with $f(\omega) = f'(\omega) - 1 = 0$, we wish to give the following:
Definition A: *For $-1 \leq \alpha < 1$, $-1 \leq \gamma < 1$, $\beta \geq 0$ and ω is an arbitrary fixed point in U, we let $\Phi_{\gamma,\omega}(f, g, \alpha, \beta, \lambda, l)$ be the subclass of $S(\omega)$ consisting of functions of the form (1) and the functions $g(z)$ of the form (2) and satisfying the analytic condition:*

$$Re\left\{\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'} - \alpha\right\} \tag{4}$$

$$> \beta\left|\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'} - 1\right|.$$

Next, we let $T(\omega)$ denote the subclass of $S(\omega)$ consisting of functions of the form

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k \qquad (a_k \geq 0). \tag{5}$$

Further we define the class $\Phi_{\gamma,\omega}T(f, g, \alpha, \beta, \lambda, l)$ by

$$\Phi_{\gamma,\omega}T(f, g, \alpha, \beta, \lambda, l) = \Phi_{\gamma,\omega}(f, g, \alpha, \beta, \lambda, l) \cap T(\omega) \tag{6}$$

which is the class of analytic functions with negative coefficients.
It is clearly seen from our definition that the classes studied by Subramanian [18], Bharati et al [6], Murugusundaramothy and Magesh [10,11], Rosy and Murugusundaramothy [15], Shams et al [17], Aouf and Moustafa [3], Murugusundaramothy et al [12], Ahuja et al [2], Aouf et al [4] and many other new ones could be derived with special choices of the parameter involved.
Also, $I^n(\lambda, l)$ was introduced and studied by Aouf et al [5]. That is,
$I^0(\lambda, l) : A(\omega) \to A(\omega)$ as follows
$I^0(\lambda, l)f(z) = f(z)$

$$\begin{aligned} I^1(\lambda,l)f(z) = I(\lambda,l)f(z) &= I^0(\lambda,l)f(z)\frac{1-\lambda+l}{1+l} + (I^0(\lambda,l)f(z))'\frac{\lambda(z-\omega)}{1+l} \\ &= (z-\omega) + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)a_k(z-\omega)^k \end{aligned}$$

and

$$I^2(\lambda,l)f(z) = I^1(\lambda,l)f(z)\frac{1-\lambda+l}{1+l} + (I^1(\lambda,l)f(z))'\frac{\lambda(z-\omega)}{1+l}$$
$$= z + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k(z-\omega)^k$$

and in general,

$$I^n(\lambda,l)f(z) = I(\lambda,l)(I^{n-1}(\lambda,l)f(z))$$
$$= (z-\omega) + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^n a_k(z-\omega)^k, n\in N_0, \lambda\geq 0, l\geq 0[5].$$

2. Coefficient Inequalities.

Theorem A: *A function $f(z)$ of the form (5) is in $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ if*

$$\sum_{k=2}^{\infty}\Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]|a_k|b_k \leq 1-\alpha \quad (7)$$

where $-1\leq\alpha<1$, $-1\leq\gamma<1$ *and* $\beta\geq 0$ *and*

$$\Psi_k^m(\lambda,l) = \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m$$

Proof: It suffices to show that

$$\beta\left|\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'} - 1\right|$$
$$-Re\left\{\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'} - 1\right\} \leq 1-\alpha.$$

We have

$$\beta\left|\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'(z)} - 1\right|$$
$$-Re\left\{\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]' + \gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right] + \gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'} - 1\right\}$$

$$\leq 1+\beta\left|\frac{(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'+\gamma(z-\omega)^2\left[I_\omega^m(\lambda,l)(f*g)(z)\right]''}{(1-\gamma)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]+\gamma(z-\omega)\left[I_\omega^m(\lambda,l)(f*g)(z)\right]'}-1\right|$$

$$\leq\frac{(1+\beta)\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(k-1)\left[1+(\gamma(k-1)\right]|a_k|b_k}{1-\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(k-1)\left[1+(\gamma(k-1)\right]|a_k|b_k}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]|a_k|b_k\leq 1-\alpha$$

and if we put$\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m=\Psi_k^m(\lambda,l)$ the proof is complete.

Theorem B: *A necessary and sufficient condition for $f(z)$ of the form (5) to be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ is that*

$$\sum_{k=2}^{\infty}\Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]a_kb_k\leq 1-\alpha \quad (8)$$

where

$$\Psi_k^m(\lambda,l)=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m$$

Proof: In view of Theorem A, we need only to prove the necessity. If $f(z)\in\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ and $z-\omega$ is real, then

$$\frac{1-\sum_{k=2}^{\infty}k\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[1+(\gamma(k-1)\right]a_kb_k(z-\omega)^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[1+(\gamma(k-1)\right]a_kb_k(z-\omega)^{k-1}}-\alpha\geq$$

$$\beta\left|\frac{1-\sum_{k=2}^{\infty}(k-1)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[1+(\gamma(k-1)\right]a_kb_k(z-\omega)^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[1+(\gamma(k-1)\right]a_kb_k(z-\omega)^{k-1}}\right|$$

Letting $(z-\omega)\to(r+d)$ along the real axis, we obtain the desired inequality.

Corollary A: *Let the function $f(z)$ be defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then*

$$a_k\leq\frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}\qquad k\geq 2 \quad (9)$$

where $\Psi_k^m(\lambda,l)=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m$. The result is sharp for the function:
$f(z)=(z-\omega)-\frac{1-\alpha}{(r+d)^{k-1}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m[k(1+\beta)-(\alpha+\beta)][1+(\gamma(k-1)]b_k}(z-\omega)^k\ k\geq 2$

3. Distortion Theorems.

Theorem C: *Let the function $f(z)$ be defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then for $|z-\omega| = r+d < 1$, $|\omega| = d$, [19], we have that*

$$|f(z)| \geq (r+d) - \frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2}(r+d) \tag{10}$$

and

$$|f(z)| \leq (r+d) + \frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2}(r+d), \tag{11}$$

provided that $b_k \geq b_2 (k \geq 2)$. The inequalities in (11) and (12) are attained for the function $f(z)$ given by

$$f(z) = (z-\omega) - \frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)(r+d)b_2}(z-\omega)^2 \tag{12}$$

where $\Psi_2^m(\lambda,l) = \left(\frac{1+\lambda+l}{1+l}\right)^m$

Proof: Since $k \geq 2$

$$\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)(r+d)b_2 \leq \Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)\right] \cdot \left[1+(\gamma(k-1)\right]b_k.$$

Using Theorem B, we have

$$\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)(r+d)b_2\sum_{k=2}^{\infty}a_k \tag{13}$$

$$\leq (r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k \leq 1-\alpha$$

that is,

$$\sum_{k=2}^{\infty}a_k \leq \frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)(r+d)b_2}. \tag{14}$$

From (5) and (15), we have

$$|f(z)| \geq (r+d)-(r+d)^2\sum_{k=2}^{\infty}a_k \geq (r+d)-\frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2}(r+d); \tag{15}$$

and

$$|f(z)| \geq (r+d)+(r+d)^2\sum_{k=2}^{\infty} a_k \leq (r+d)+\frac{1-\alpha}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2}(r+d) \tag{16}$$

This complete the proof of Theorem C.

Theorem D: *Let the function $f(z)$ defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then for $|z-\omega| = r+d < 1$, we have*

$$|f'(z)| \geq 1 - \frac{2(1-\alpha)}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2} \tag{17}$$

and

$$|f'(z)| \leq 1 + \frac{2(1-\alpha)}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2} \tag{18}$$

provided that $b_k \geq b_2 (k \geq 2)$. The result is sharp for the function $f(z)$ given by (13).

Proof: From Theorem B and (15), we have

$$\sum_{k=2}^{\infty} ka_k \leq \frac{2(1-\alpha)}{\Psi_2^m(\lambda,l)(2+\beta-\alpha)(1+\gamma)b_2}. \tag{19}$$

We omit the rest of the proof because it is similar to the proof of Theorem C.

4. Convex Linear Combination.

Theorem E: *Let $\rho_n \geq 0$ for $n = 1,2,...y$ and*

$$\sum_{n=1}^{y} \rho_n \leq 1.$$

If the functions $F_n(z)$ defined by

$$F_n(z) = (z-\omega) - \sum_{k=2}^{\infty} a_{k,n}(z-\omega)^k \quad (a_k \geq 0, n = 1,2,...y) \tag{20}$$

are in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ for every $n = 1,2,...y$, then the function $f(z)$ defined by

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty}\left(\sum_{n=1}^{y} \rho_n a_{k,n}\right)(z-\omega)^k$$

is in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$.
Proof: Since $F_n(z) \in \Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$, it follows from Theorem B that

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]a_{k,n}b_k \le 1-\alpha \quad (21)$$

for every $n = 1, 2, ...y$. Hence

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]\left(\sum_{n=1}^{y}\rho_n a_{k,n}\right)b_k =$$

$$\sum_{n=1}^{y}\rho_n\left(\sum_{k=2}^{\infty}(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]a_{k,n}b_k\right)$$
$$\le (1-\alpha)\sum_{n=1}^{y}\rho_n \le 1-\alpha.$$

By Theorem B, it follows that $f(z) \in \Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$
Corollary B: *The class* $\Phi_{\gamma,\omega}(f,g,\alpha,\beta)$ *is closed under convex linear combinations.*
Theorem F:*Let* $f_1(z) = (z-\omega)$ *and*

$$f_k(z) = (z-\omega) - \frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}(z-\omega)^k; \quad (22)$$

for $k \ge 2$, $-1 \le \alpha < 1$, $0 \le \gamma \le 1$, *and* $\beta \ge 0$. *Then* $f(z)$ *is in the class* $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty}\rho_k f_k(z), \quad (23)$$

where $\rho_k \ge 0$ *and* $\sum_{k=1}^{\infty}\rho_k = 1$.
Proof: Assume that

$$f(z) = \sum_{k=1}^{\infty}\rho_k f_k(z) = \quad (24)$$

$$(z-\omega) - \sum_{k=2}^{\infty}\frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}\rho_k(z-\omega)^k$$

Then it follows that

$$\frac{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}{1-\alpha} \tag{25}$$

$$\times\frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}\rho_k=$$

$$=\sum_{k=2}^{\infty}\rho_k=1-\rho_1\leq 1.$$

So, by Theorem B $f(z)\in\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$.
For the converse, we assume that the function $f(z)$ defined by (5) belongs to the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then

$$a_k\leq\frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}\quad k\geq 2. \tag{26}$$

On setting

$$\rho_k=\frac{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]a_kb_k}{1-\alpha}\quad k\geq 2 \tag{27}$$

and

$$\rho_1=1-\sum_{k=2}^{\infty}\rho_k \tag{28}$$

this shows that $f(z)$ can be expressed in the form (24) and the proof is complete.
Corollary C: The extreme points of the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ are the functions $f_1(z)=(z-\omega)$ and

$$f_k(z)=(z-\omega)-\frac{1-\alpha}{(r+d)^{k-1}\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}(z-\omega)^k \tag{29}$$

for $k\geq 2$.

5. Radii of ω-close-to-convexity, ω-starlikeness and ω-convexity.

Theorem G: *Let the function $f(z)$ defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then $f(z)$ is $\omega-$close$-$to$-$convex of order $\lambda(0\leq\lambda<1)$ in $|z-\omega|<\Gamma_1$, where*

$$\Gamma_1=\inf_{k\geq 2}\left\{\frac{(1-\lambda)\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}b_k}{k(1-\alpha)}\right\}^{\frac{1}{k-1}} \tag{30}$$

The result is sharp, the extremal function being given by (10).
Proof: We have to show that $|f'(z)-1|\leq 1-\lambda \quad for\,|z-\omega|<\Gamma_1$ where Γ_1 is given by (31). Indeed we find from definition (5) that

$$|f'(z)-1|\leq\sum_{k=2}^{\infty}ka_k\left|z-\omega\right|^{k-1}$$

Thus $|f'(z)-1|\leq 1-\lambda$ if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\lambda}\right)a_k\left|z-\omega\right|^{k-1}\leq 1. \tag{31}$$

By Theorem B, (32) will be true if

$$\left(\frac{k}{1-\lambda}\right)\left|z-\omega\right|^{k-1}\leq\frac{\Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)-2\right]\left[1+(\gamma(k-1)\right]b_k}{(1-\alpha)}$$

that is, if

$$|z-\omega|\leq\left\{\frac{(1-\lambda)\Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]b_k}{k(1-\alpha)}\right\}^{\frac{1}{k-1}} \tag{32}$$

for $k\geq 2$. Theorem G follows easily from (33).
Theorem H:*Let the function $f(z)$ defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$. Then $f(z)$ is $\omega-$ starlike of order $\lambda(0\leq\lambda<1)$ in $|z-\omega|<\Gamma_2$, where*

$$\Gamma_2=\inf_{k\geq 2}\left\{\frac{(1-\lambda)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]\Psi_k^m(\lambda,l)(r+d)^{k-1}b_k}{(k-\lambda)(1-\alpha)}\right\}^{\frac{1}{k-1}} \tag{33}$$

The result is sharp, with the extremal function $f(z)$ given by (10).
Proof: It is sufficient to show that

$$\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right|\leq 1-\lambda \qquad for\,|z-\omega|<\Gamma_2,$$

where Γ_2 is given by (34);. We find again from the definition (5) that

$$\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k\left|z-\omega\right|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\left|z-\omega\right|^{k-1}}.$$

Thus $\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| \leq 1-\lambda$ if

$$\sum_{k=2}^{\infty}\frac{(k-\lambda)a_k\left|z-\omega\right|^{k-1}}{1-\lambda} \leq 1. \tag{34}$$

But, by Theorem B (35) will be true if

$$\frac{(k-\lambda)\left|z-\omega\right|^{k-1}}{1-\lambda} \leq \frac{\Psi_k^m(\lambda,l)(r+d)^{k-1}}{1-\alpha}.$$

$$\cdot\frac{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}b_k}{1-\alpha}$$

that is, if

$$|z-\omega| \leq \left\{\frac{(1-\lambda)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]\Psi_k^m(\lambda,l)(r+d)^{k-1}b_k}{(k-\lambda)(1-\alpha)}\right\}^{\frac{1}{k-1}}, \tag{35}$$

for $k \geq 2$.

Theorem H follows easily from (36).

Corollary D:*Let the function $f(z)$ defined by (5) be in the class $\Phi_{\gamma,\omega}(f,g,\alpha,\beta)$. Then $f(z)$ is $\omega-convex$ of order $\lambda(0 \leq \lambda < 1)$ in $|z-\omega| < \Gamma_3$ where*

$$\Gamma_3 = \inf_{k\geq 2}\left\{\frac{(1-\lambda)\left[k(1+\beta)-(\alpha+\beta)-\right]\left[1+(\gamma(k-1)\right]\Psi_k^m(\lambda,l)(r+d)^{k-1}b_k}{k(k-\lambda)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{36}$$

The result is sharp, with the extremal function $f(z)$ given by (10).

6. A family of integral operator

In view of Theorem B, we see that $(z-\omega)-\sum_{k=2}^{\infty}d_k(z-\omega)^k$ is in $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ as long as $0 \leq d_k \leq a_k$ for all k. In particular, we have

Theorem I: *Let the function $f(z)$ defined by (5) be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$ and σ be a real number such that $\sigma > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{\sigma+1}{(z-\omega)^{\sigma}} \int_{\omega}^{z} (t-\omega)^{\sigma-1} f(t)dt(\sigma > -1) \tag{37}$$

also belongs to the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$
Proof: From representation (38) of $F(z)$, it follows that

$$F(z) = (z-\omega) - \sum_{k=2}^{\infty} d_k (z-\omega)^k,$$

where $d_k = \left(\frac{\sigma+1}{\sigma+k}\right) a_k \leq a_k \qquad (k \geq 2)$ The converse is not true and this leads to a radius of univalence result.
Theorem J: *Let the function $F(z)= (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k \qquad (a_k \geq 0)$ be in the class $\Phi_{\gamma,\omega}T(f,g,\alpha,\beta,\lambda,l)$, and let σ be real number such that $\sigma > -1$. Then the function $f(z)$ given by (38) is univalent in $|z-\omega| < R^*$, where*

$$R^* = \inf_{k\geq 2} \left\{ \frac{(\sigma+1)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right]\Psi_k^m(\lambda,l)(r+d)^{k-1}b_k}{k(\sigma+k)(1-\alpha)} \right\}^{\frac{1}{k-1}} \tag{38}$$

The result is sharp.
Proof: From (38), we have

$$f(z) = \frac{(z-\omega)^{(1-\sigma)}\left[(z-\omega)^{\sigma}F(z)\right]'}{(\sigma+1)} = (z-\omega) - \sum_{k=2}^{\infty} \left(\frac{\sigma+k}{\sigma+1}\right) a_k (z-\omega)^k (\sigma \geq -1)$$

In order to obtain the required result, it suffices to show that

$$|f'(z)-1| < 1 \qquad whenever\, |z-\omega| < R^*$$

where R^* is given by (39). Now $|f'(z)-1| \leq \sum_{k=2}^{\infty} \frac{k(\sigma+k)}{\sigma+1} a_k |z-\omega|^{k-1}$ Thus $|f'(z)-1| < 1$ if

$$\sum_{k=2}^{\infty} \frac{k(\sigma+k)}{\sigma+1} a_k |z-\omega|^{k-1} < 1 \tag{39}$$

But Theorem B confirms that

$$\sum_{k=2}^{\infty} \frac{\Psi_k^m(\lambda,l)(r+d)^{k-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}a_k b_k}{(1-\alpha)} \leq 1 \tag{40}$$

Hence (40) will be satisfied if

$$\frac{k(\sigma+k)}{\sigma+1}\left|z-\omega\right|^{k-1} < \frac{\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}b_k}{(1-\alpha)}$$

that is if,

$$|z-\omega|\left\{\frac{(\sigma+1)\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}b_k}{k(\sigma+k)(1-\alpha)}\right\}^{\frac{1}{k-1}}$$

Therefore, the function $f(z)$ given by (38) is univalent in $|z-\omega|<R^*$. sharpness of the result follows if we take

$$f(z)=(z-\omega)-\frac{(\sigma+k)(1-\alpha)}{\Psi_k^m(\lambda,l)\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(\gamma(k-1)\right](r+d)^{k-1}b_k(\sigma+1)}(z-\omega)^k,$$

for $k\geq 1$.

References

[1] M. Acu and S. Owa (2005): On some subclasses of univalent functions, Journal of inequalities in Pure and Applied Mathematics Vol, 6 issue 3, Article 70, 1 - 14 2005

[2] O.P. Ahuja, G. Murugusundaramorthy, N. Magesh. Integral means for uniformly convex and starlike functions associated with generalized hypergeometric functions. JIPAM 8(4) Article 118, 9, 2007.

[3] M.K. Aouf and A.O. Mostafa: Some properties of a subclass of uniformly convex functions with negative coefficients. Demonstratio Math. 41(2) : 353-370, 2008.

[4] M.K. Aouf, R.M. El-Ashwah, and S.M. El-Deeb: Certain subclasses of uniformily starlike and convex functions defined by convolution. Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 26, 55-70 (2010).

[5] M.K Aouf, A. Shamandy, A.O. Mostafa and M.Media: A subclass of $M-\omega$ starlike function. Acta Universitatis Apulensis 23, 135-142 2010.

[6] R. Bharati, R. Parvatham and A. Swaminathan: On subclasses of uniformly convex functions and corresponding class of starlike function. Tamkang.J. Math, 28(1):17-32 1997.

[7] A.W. Goodman: On uniformly convex functions. Ann.polon. Math 56(1): 87-92 1991

[8] A.W. Goodman: On uniformly starlike functions. J. Math. Anal. Appl. 155(2): 364-370 1991

[9] S. Kanas and F. Ronning (1999): Uniformly starlike and convex functions and other related classes of univalent functions. Ann. UNiv. Mariae Curie - Sklodowska Section A, 53, 95-105.

[10] G. Murugusundaramorthy and N. Magesh: A new subclass of uniformily convex functions and a corresponding subclass of stalike functions with fixed second coefficient. J. Ineq. Pure and Appl. Math. 5(4) Art; 85, 10pp. (electronic), 2004

[11] G. Murugusundaramorthy and N. Magesh: Linear operators associated with a subclass of uniformly convex functions. IJPAMS 3(1): 113-125, 2006.

[12] G. Murugusundaramorthy and N. J. Rosy and Muthunagai Carlson-shaffer operator and their applications to certain subclass of uniformly convex functions. Gen. Math., 15(4): 131-143, 2007.

[13] A.T. Oladipo: On a subclasses of univalent functions Advances in Applied Mathematical Analysis Vol 4 No 2: 87-93 (2009).

[14] A.T. Oladipo: On certain subclasses of analytic and univalent functions involving convolution operators. Acta Universitatis No. 20, 163-174, 2009.

[15] T. Rosy and G. Murugusundaramorthy: Fractional calculus and their applications to certain subclass of uniformly convex functions. Far East J. Math. Sci (FJMS), 15(2): 231-242,2004.

[16] S.Ruscheweyh: Linear operators between classes of prestarlike functions. Comment. Math. Nelv., 52: 497-509.

[17] S.Shams, S.R. Kulkarni and J.M. Jahangiri: Classes of uniformly starlike and convex functions. Int. J. Math. Math. sci (53-56): 2959-2961, 2004.

[18] K.G. Subramanian, G. Murugusundaramorthy, P. Bala subrahmanyan and H. Silverman. Subclasses of uniformly convex and uniformly starlike functions. Math. Japon., 42(3): 517-522, 1995.

[19] J.K. Wald: On starlike functions, ph.D Thesis, University of Delaware, New Ark, Delaware 1978.

Abiodun Tinuoye Oladipo
Department of Pure and Applied Mathematics,
Ladoke Akintola University of Technology, Ogbomoso
P. M. B. 4000, Ogbomoso, Nigeria.
email:*atlab_3@yahoo.com*