ESTIMATION OF NORMALIZED EIGENFUNCTIONS OF SPECTRAL PROBLEM WITH SMOOTH COEFFICIENTS

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Abstract. In this work , we studies asymptotic behaviors of eigen values in both cases regular and irregular and also, estimation of normalized eigenfunctions to the spectral problem:

$$-y'' + q(x)y = \lambda^2 p(x)y, x \in [0,a]\,, y'(0) = 0,\ y'(a) + i\lambda y(a) = 0,$$

$$\left(\int_0^a p(x)\,|y(x)|^2\right)^{\frac{1}{2}}.$$

Where λ is spectral parameter and $q(x)$ and $\rho(x)$ are smooth and $q(x) \in C\,[0,a]$ and $\rho(x) \in C^2\,[0,a]$.

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1. Introduction

Though by the present time many spectral problems has been studied well enough [1], [6], and [9-11] and it is possible to consider their general theory the problems arising in modern completed, however application of this theory to specific problems in some cases is inconvenient. Therefore analysis of such problems is actual. Besides many classical results have been received at very rigid restrictions on smoothness of coefficients while the coefficients of applications, mostly, do not satisfy the demanded conditions of smoothness. Besides many classical results are generally incorrect. In the present work, we studies asymptotic behavior and estimated the normalized eigenfunction to the second order boundary value problem of the type (1) - (3) whenever the coefficients are smooth functions. Consider the spectral problem:

$$-y'' + q(x)y = \lambda^2 p(x)y, x \in [0, a], \tag{1}$$

$$y'(0) = 0,\ y'(a) + i\lambda y(a) = 0, \tag{2}$$

$$\left(\int_0^a p(x) |y(x)|^2\right)^{\frac{1}{2}}. \tag{3}$$

Where λ is spectral parameter.

Aigounov and Tamila [5], studied estimation of normalized eigenfunction to the T.Regge problem of the type : $-y'' + q(x)y = \lambda^2 p(x)y, x \in [0, a]$, $y'(0) = 0,\ y'(a) + i\lambda y(a) = 0$, $\left(\int_0^a p(x) |y(x)|^2\right)^{\frac{1}{2}}$ where the coefficients$q(x)$ and $\rho(x)$ are smooth and $q(x) \in C[0, a]$ and $\rho(x) \in C^2[0, a]$ and they proved , that normalized eigenfunctions of the T.Regge problem are uniformly bounded, if $\rho(a) \neq 1$ (regular case) and graw as $a.\sqrt{|\lambda|}$ in the case of $\rho(a) = 1$ (irregular case) and a is constant. But the present work, we shows the estimation of normalized eigenfunctions of the spectral problem (1) - (3) are uniformly bounded in both cases regular and irregular where the coefficients are smoothes . Jwamer and Aigounov [7] studied the estimation of normalized eigenfunction to the T.Regge problem for non smooth coefficients. Also, Jwamer and Khelan [8] showed the same result in work Tamila [13] but for different boundary value problem and whenever the coefficients $q(x)$ and $\rho(x)$ are constants. Some Arthurs worked on the some different models of spectral problem to the types of linear and nonlinear differential equations, for more information about the results which they obtained see [2-3].

2. Asymptotic behaviors of eigen values to the spectral problem (1)-(3) in both cases regular and irregular

In this section , first we proves some important lemma and theorems, which they helps us to studies asymptotic behaviors of eigen values to our spectral problem Assume, that the numbering of the roots of w_k $\left(w_k = \sqrt{1}\right)$ is given by

$$\operatorname{Re}(iw_0) \leq \operatorname{Re}(iw_1\lambda) \tag{4}$$

Entire complex plane of $\lambda = \delta + i\sigma$ can be divided in to 4 sectors with vertex at $\lambda = 0$, so that each sector T_k different roots of w_k can be ordered so that for $\lambda \in T_k$ satisfy the inequality (4). Sector T_k in the plane ? are determined by the inequalities $\frac{(k-1)\pi}{2} \leq \arg\lambda \leq \frac{kp}{2}, k = \overline{1,4}$. The following lemma holds:

Lemma 1 *Suppose $\lambda \in T_k, k = \overline{1,4}$ and w_k satisfy (4). then there are two linearly independent solution $y_k(x,\lambda)$ of equation (1), regular for sufficiently large $|\lambda|$ and such, that when $s = 0,1$ uniformly in $0 \leq x \leq a$ we have:*

$$y_k^s(x,\lambda) = (\phi_k\lambda)^s e^{\lambda\int_0^x \phi_k dt}\left[A_0 + \frac{A_{1s}}{\lambda} + \cdots + \frac{A_{ns}}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right], \tag{5}$$

where

$$A_0 = \frac{1}{(\rho(x))^{\frac{1}{4}}}, A_{1s} = A_1 + \frac{A_0^{'}}{\phi_k}C_s^1 + \frac{A_0\phi_k^{'}}{\phi_k^2}C_s^2,$$

$$\vdots$$

$$A_{ns} = A_n + \frac{A_{n-1}^{'}}{\phi_k}C_s^1 + \frac{A_{n-1}\phi_k^{'}}{\phi_k^2}C_s^2 + \frac{A_{n-2}^{''}}{\phi_k^2}C_s^2,$$

$$A_1 = \frac{1}{2iw_k}A_0\int_0^x \left(q(t)A_0 - A_0^{''}\right)A_0 dt,$$

$$\vdots$$

$$A_n = \frac{1}{2iw_k}A_0\int_0^x \left(q(t)A_{n-1} - A_{n-1}^{''}\right)A_0 dt.$$

Proof: According Ya.D.Tamarkin[12], there exists a fundamental system of differentiable solutions of (1), the asymptotic behavior of which at $|\lambda| \to \infty$ is given by $y_k(x,\lambda) \sim e^{\lambda\int_0^x \phi_k dt}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}, (k = 0,1)$.
We seek the solution of equation (1) as

$$y_k(x,\lambda) = e^{\lambda\int_0^x \phi_k dt}\left[A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + \cdots + \frac{A_n}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right].$$

We find $y_k^{'}(x,\lambda)$ and $y_k^{''}(x,\lambda)$, and substituting these relations in equation(1), we obtain:

$$-e^{\lambda\int_0^x \phi_k dt}(\lambda\phi_k)^2\left[A_0+\frac{1}{\lambda}\left(A_1+\frac{2A_0'}{\phi_k}+\frac{\phi_k'}{\phi_k^2}A_0\right)\right.$$
$$+\frac{1}{\lambda^2}\left(A_2+\frac{2A_1'}{\phi_k}+\frac{\phi_k'A_1}{\phi_k^2}+2\frac{A_0'\phi_k'}{\phi_k^3}+\frac{A_0''}{\phi_k^2}\right)+\dots$$
$$\left.+\frac{1}{\lambda^n}\left(A_n+\frac{2A_{n-1}'}{\phi_k}+\frac{A_{n-1}\phi_k'}{\phi_k^2}+2\frac{A_{n-2}'\phi_k'}{\phi_k^3}+\frac{A_{n-2}''}{\phi_k^2}\right)+O\left(\frac{1}{\lambda^{n+1}}\right)\right]$$
$$+q(x)e^{\lambda\int 0^x\phi_k dt}\left[A_0+\frac{A_1}{\lambda}+\frac{A_2}{\lambda^2}+\dots+\frac{A_n}{\lambda^n}+O\left(\frac{1}{\lambda^{n+1}}\right)\right]$$
$$+\lambda^2\rho(x)e^{\lambda\int 0^x\phi_k dt}\left[A_0+\frac{A_1}{\lambda}+\frac{A_2}{\lambda^2}+\dots+\frac{A_n}{\lambda^n}+O\left(\frac{1}{\lambda^{n+1}}\right)\right]$$

Dividing by $\lambda^2\rho(x).e^{\lambda\int_0^x\phi_k dt}$ and equaling the coefficients of equal powers of λ, we obtain:
$\lambda^{-1}: 2A_0'+\frac{\phi_k'}{\phi_k}A_0=0,$
$\lambda^{-2}: 2A_1'+\frac{\phi_k'}{\phi_k}A_1=q(x)\frac{A_0}{\phi_k}-\frac{(A_0''}{\phi_k},$
$\vdots$
$\lambda^{-(n+1)}: 2A_n'+\frac{\phi_k'}{\phi_k}A_n=q(x)\frac{A_{n-1}}{\phi_k}-\frac{A_{n-1}''}{\phi_k},$
Solving these equations, we get:
$A_0=\frac{1}{(\rho(x))^{\frac{1}{4}}};$
$A_1=\frac{1}{2iw_k}A_0\int_0^x(q(t)A_0-A_0'')A_0dt;$
$\vdots$
$A_n=\frac{1}{2iw_k}A_0\int_0^x(q(t)A_{n-1}-A_{n-1}'')A_0dt.$
Which proves the lemma.

Theorem 1 *Asymptotic of eigen values for problem (1)-(3) in the case of regular with and in the sector T_1 has the form:*

$$\lambda_m=\frac{1}{d}\left(m\pi-\frac{i}{2}lnC_0+O\left(\frac{1}{m}\right)\right)$$

and in the sector T_2 asymptotic of spectral has the form:

$$\lambda_m=\frac{1}{d}\left(m\pi+\frac{i}{2}lnC_0+O\left(\frac{1}{m}\right)\right)$$

where $C_0 = \left(\frac{\sqrt{\rho(a)}-1}{1+\sqrt{\rho(a)}}\right)$

Proof: Consider the determinant of $\Delta(\lambda)$, defined by

$$\Delta(\lambda) = |U_k(y_j)|_{k,j=0,1}, \tag{6}$$

where $U_0(\tilde{y}_j) = \tilde{y}_j'(0) = 0 j = 0,1$, $U_1(\tilde{y}_j) = (-i\lambda w_k)\tilde{y}_j(a,\lambda) + \tilde{y}_j'(a,\lambda) = 0$, $U_1(\tilde{y}_j) = i\lambda\tilde{y}_j(a,\lambda) + \tilde{y}_j'(a,\lambda) = 0$.
If in eq.(5), we take only two elements as follows

$$y_k^{(s)}(x,\lambda) = (\phi_k\lambda)^s e^{\lambda\int_0^x \phi_k dx}\left[A_0 + O\left(\frac{1}{\lambda}\right)\right], s = 0,1 \tag{7}$$

Using the formula (7) and the boundary condition we obtain:

$$U_0(\tilde{y}_j) = (iw_j'\lambda\sqrt{\rho(0)})\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right], where w_j' = e^{(i\frac{(j-k)\pi}{n}}$$

$$U_0(\tilde{y}_0) = (iw_0'\lambda\sqrt{\rho(0)})\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right] = (i\lambda\sqrt{\rho(0)})\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right],$$

$$U_0(\tilde{y}_1) = (iw_1'\lambda\sqrt{\rho(0)})\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right] = (i\lambda\sqrt{\rho(0)})\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right],$$

$$U_1(\tilde{y}_0) = i\lambda e^{-i\lambda d}\left(1 - \sqrt{\rho(a)}\right)\left[\frac{1}{(\rho(a))^{\frac{1}{4}}}\right],$$

$$U_1(\tilde{y}_1) = i\lambda e^{i\lambda d}\left(1 + \sqrt{\rho(a)}\right)\left[\frac{1}{(\rho(a))^{\frac{1}{4}}}\right],$$

Substituting each of them in $\Delta(\lambda) = 0$, and we obtain

$$\Delta(\lambda) = \begin{vmatrix} i\lambda\sqrt{\rho(0)}\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right] & i\lambda\sqrt{\rho(0)}\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right] \\ i\lambda e^{-i\lambda d}\left(1 - \sqrt{\rho(a)}\right)\left[\frac{1}{(\rho(a))^{\frac{1}{4}}}\right] & i\lambda e^{i\lambda d}\left(1 + \sqrt{\rho(a)}\right)\left[\frac{1}{(\rho(a))^{\frac{1}{4}}}\right] \end{vmatrix} = 0$$

Suppose $f(\lambda) = (i\lambda)^2\sqrt{\rho(0)}\left[\frac{1}{(\rho(0))^{\frac{1}{4}}}\right]\left[\frac{1}{(\rho(a))^{\frac{1}{4}}}\right]$
$\rightarrow \Delta(\lambda) = f(\lambda)\left[\left(1+\sqrt{\rho(a)}\right)e^{i\lambda d} + \left(1-\sqrt{\rho(a)}\right)e^{-i\lambda d}\right] = 0$
$\rightarrow e^{2i\lambda d} = \frac{\sqrt{\rho(a)}-1}{1+\sqrt{\rho(a)}} = C_0$
$\rightarrow e^{2i\lambda d} = C_0 \rightarrow 2i\lambda d = lnC_0 + 2m\pi i + O\left(\frac{1}{m}\right)$
$\rightarrow \lambda_m = \frac{1}{d}\left(m\pi - \frac{i}{2}lnC_0 + O\left(\frac{1}{m}\right)\right)$
In the case of regular the sector T_1 asymptotic of spectrum has the form:
$\lambda_m = \frac{1}{d}\left(m\pi - \frac{i}{2}lnC_0 + O\left(\frac{1}{m}\right)\right)$
And in the sector T_2 $\lambda_m = \frac{1}{d}\left(m\pi + \frac{i}{2}lnC_0 + O\left(\frac{1}{m}\right)\right)$
This complets the proof of Theorem.

Theorem 2 *Asymptotic of eigenvalues for problem (1)-(2) in the case of regular with* $\rho(a) = 1$, $\rho'(a) = 0$, $q(a) + \frac{1}{4}\rho''(a) \neq 0$, *and in the sector* T_1 *has the form:*
$\lambda_m = \frac{1}{d}\left(-m\pi + \frac{i}{2}lnC_0 + \frac{i}{2}ln\lambda^2\right) + \bar{\bar{o}}(1)$
and in the sector T_2 *asymptotic of spectrum has the form:*
$\lambda_m = \frac{1}{d}\left(-m\pi - \frac{i}{2}lnC_0 - \frac{i}{2}ln\lambda^2\right) + \bar{\bar{o}}(1)$
where $C_0 = \frac{(2i)^2}{q(a)+\frac{1}{4}\rho''(a)}$, *Such that* $q(a) + \frac{1}{4}\rho''(a) \neq 0$

Proof: From the boundary conditions (5), we have

$$U_0(y_0) = y_0'(0) = i\lambda\sqrt{\rho}\left(\rho^{\frac{-1}{4}}(0) + O\left(\frac{1}{\lambda^2}\right)\right)$$

$$U_0(y_1) = y_1'(0) = -i\lambda\sqrt{\rho}\left(\rho^{\frac{-1}{4}}(0) + O\left(\frac{1}{\lambda^2}\right)\right)$$

$$U_1(y_0) = e^{i\lambda d}(i\lambda)\left(\left(\sqrt{\rho(a)}+1\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right)\right.$$
$$\left.+2\frac{1}{\lambda}\frac{1}{2i}A_{0(p)}'(a) + \frac{1}{\lambda^2}\frac{1}{(2i)^2}A_{1(p)}'(a) + O\left(\frac{1}{\lambda^2}\right)\right)$$

$$U_1(y_1) = e^{-i\lambda d}(i\lambda)\left(\left(1-\sqrt{\rho(a)}\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right)\right.$$
$$\left.-2\frac{1}{\lambda}\frac{1}{-2i}A_{0(p)}'(a) - 2\frac{1}{\lambda^2}\frac{1}{(-2i)^2}A_{1(p)}'(a) + O\left(\frac{1}{\lambda^2}\right)\right)$$

Where $A_{0(p)}(a) = 1$,

$$A_{1(p)}(a) = \int_0^a (q(t)A_0 - A_0^{''})A_0 dt;$$
$$\vdots A_{n(p)}(a) = \int_0^a (q(t)A_{n-1} - A_{n-1}^{''})A_0 dt;$$

Spectrum of (1)-(2) coincides with the set of roots of the equation: $\Delta(\lambda) = 0$, then

$$\Delta(\lambda) = \begin{vmatrix} i\lambda\sqrt{\rho}\left[\rho^{\frac{-1}{4}}(0) + O\left(\frac{1}{\lambda^2}\right)\right] & -i\lambda\sqrt{\rho}\left[\rho^{\frac{-1}{4}}(0) + O\left(\frac{1}{\lambda^2}\right)\right] \\ e^{-i\lambda d}(i\lambda)C & e^{-i\lambda d}(i\lambda)D \end{vmatrix} = 0$$

where

$$C = \left(\left(\sqrt{\rho(a)} + 1\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right)\right.$$
$$\left. + 2\frac{1}{\lambda}\frac{1}{2i}A_{0(p)}^{'}(a) + \frac{1}{\lambda^2}\frac{1}{(2i)^2}A_{1(p)}^{'}(a) + O\left(\frac{1}{\lambda^2}\right)\right)$$
$$D = \left(\left(1 - \sqrt{\rho(a)}\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right)\right.$$
$$\left. - 2\frac{1}{\lambda}\frac{1}{-2i}A_{0(p)}^{'}(a) - 2\frac{1}{\lambda^2}\frac{1}{(-2i)^2}A_{1(p)}^{'}(a) + O\left(\frac{1}{\lambda^2}\right)\right)$$

thus

$$e^{-2i\lambda d}[1] - \frac{(i\lambda)^2\sqrt{\rho}\left(C_1 + 2\frac{1}{\lambda}\frac{1}{2i}A_{0(p)}^{'}(a) + \frac{1}{\lambda^2}\frac{1}{(2i)^2}A_{1(p)}^{'}(a) + O\left(\frac{1}{\lambda^2}\right)\right)}{(i\lambda)^2\sqrt{\rho}\left(D_1 - 2\frac{1}{\lambda}\frac{1}{-2i}A_{0(p)}^{'}(a) - \frac{1}{\lambda^2}\frac{1}{(-2i)^2}A_{1(p)}^{'}(a) + O\left(\frac{1}{\lambda^2}\right)\right)}[1] = 0$$

$$C_1 = \left(\sqrt{\rho(a)} + 1\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right)$$
$$D_1 = \left(\sqrt{\rho(a)} - 1\right)\left(A_{0(p)}(a) + \frac{1}{\lambda}\frac{1}{2i}A_{1(p)}(a)\right).$$

From which we obtain that:

$$e^{-2i\lambda d}[1] - \frac{\left(-4\lambda^2 - 2\frac{\lambda}{i}A_1(p) + A'_{1(p)}(a)\right)}{q(a)+\frac{1}{4}\rho''(a)}[1] = 0$$
$$\to e^{-2i\lambda d}[1] = \frac{-4\lambda^2}{q(a)+\frac{1}{4}\rho''(a)}[1]$$

$$\to e^{-2i\lambda d}[1] = C_0.\lambda^2, \text{where } C_0 = \frac{-4}{q(a)+\frac{1}{4}\rho''(a)} = \frac{(2i)^2}{q(a)+\frac{1}{4}\rho''(a)},$$
$$q(a)+\frac{1}{4}\rho^{''}(a) \neq 0.$$

Taking the initial approximation $\lambda_0 = \frac{-m\pi}{d}$, and using the method of successive approximations we obtain? λ_m in the sector T_1 :

$$\lambda_m = \frac{1}{d}\left(-m\pi + \frac{i}{2}lnC_0 + \frac{i}{2}ln\lambda^2\right) + \bar{\bar{o}}(1)$$

And in the sector T_2

$$\lambda_m = \frac{1}{d}\left(-m\pi - \frac{i}{2}lnC_0 - \frac{i}{2}ln\lambda^2\right) + \bar{\bar{o}}(1)$$

This complets the proof of theorem.

3. Estimation of normalized eigenfuntions for an regular and irregular cases to the problem (1)-(3)

In this, section we obtained the estimations of normalized eigenfunctions to the problem (1)-(3), with smooth coefficients, i.e. $q(x) \in C_{[0,a]}$, $\rho(x) \in C^2_{[0,a]}$, , also proves the uniformly bounded in the both cases regular and irregular.

Theorem 3 *Suppose* $q(x) \in C_{[0,a]}$, $\rho(x) \in C^2_{[0,a]}$, *then the normalized eigenfunctions of the problem (1)-(3) in the both cases regular and irregular are uniformly bounded, i.e* $K_1 = \max_{x\in[0,a]} |y_m(x)| = K_2$, *where* K_1 *and* K_2 *does not depend on* m.

We proceed to prove theorems If the problem (1) - (3) to make a double substitution $\xi = C.\int_0^x \frac{dt}{A^2(t)}(C > 0)$ and $A^{'}(0) = 0$
$y(x) = A(x).\eta(\xi(x))$, then we obtain from the equalities
$y^{'}(x) = A^{'}(x)\eta(\xi(x)) + A(x)\eta^{'}(\xi(x))\xi^{'}(x)$ and
$y^{''}(x) = A^{''}(x)\eta(\xi(x)) + A^{'}(x)\eta^{'}(\xi(x))\xi^{'}(x) + A^{'}(x)\eta^{'}(\xi(x))\xi^{'}(x)$
$+A(x)\eta^{''}(\xi(x))(\xi^{'}(x))^2+A(x)\eta^{'}(\xi(x))\xi^{''}(x)$ $y^{''}(x) = A^{''}(x)\eta(\xi(x))+A(x)(\xi^{'}(x))^2\eta^{''}(\xi(x))$
since $2A^{'}(x)\xi^{'}(x) + A(x)\xi^{''}(x) = 0$,
$2A^{'}(x)\frac{C}{A^2(x)} - 2A(x)\frac{CA^{'}(x)}{A^3(x)} = 2A^{'}(x)\frac{C}{A^2(x)} - 2A^{'}(x)\frac{C}{A^2(x)} = 0$
$-\{A''(x)\eta(\xi(x))+A(x)(\xi^{'}(x))^2\eta''(\xi(x))\}+q(x)A(x).\eta(\xi(x)) = \lambda^2\rho(x)A(x).\eta(\xi(x))$
$A'(0)\eta(\xi(0)) + A(0)\eta'(\xi(0))\xi'(0) = 0$
$A'(0)\eta(0) + A(0)\eta'(0)\frac{C}{A^2(0)} = 0$
since $\xi(0) = C\int_0^0 \frac{dt}{A^2(t)} = 0$ and $\xi' = \frac{C}{A^2(x)}$
$\eta'(0)\frac{C}{A(0)} = 0$ since $A'(0) = 0$
$C\eta'(0) = 0$
$\eta'(0) = 0$
$A'(a)\eta(\xi(a)) + A(a)\eta'(\xi(a))\xi'(a) + i\lambda A(a).\eta(\xi(a)) = 0$
$\eta'(\xi(a)) + \frac{A'(a)\eta(\xi(a))}{A(a)\xi'(a)} + i\lambda\frac{A(a).\eta(\xi(a))}{A(a)\xi'(a)} = 0$
$\eta'(\xi(a)) + \frac{A'(a)\eta(\xi(a))}{A(a)\frac{C}{A^2(a)}} + i\lambda\frac{\eta(\xi(a))}{\frac{C}{A^2(a)}} = 0$

$$\eta'(\xi(a)) + \left(\frac{A'(a)A(a)}{C} + i\lambda\frac{A^2(a)}{C}\right)\eta(\xi(a)) = 0$$

$$\int_0^a \rho(x)A^2(x)\left|\eta(\xi(x))\right|^2 dx = 1$$

$-\{A''(x)\eta(\xi(x))+A(x)(\xi^{'}(x))^2\eta''(\xi(x))\}+q(x)A(x).\eta(\xi(x)) = \lambda^2\rho(x)A(x).\eta(\xi(x))$
Put $\xi'(x0 = \frac{C}{A^2(x)}$
$-\{A''(x)\eta(\xi(x))+A(x)\frac{C^2}{A^4(x)}\eta''(\xi(x))\}+q(x)A(x).\eta(\xi(x)) = \lambda^2\rho(x)A(x).\eta(\xi(x))$
$-\eta''(\xi) + \left(\frac{q(x)A(x)-A''(x)}{C^2}\right)A^3(x)\eta(\xi) = \frac{\lambda^2\rho(x)A^4(x)}{C^2}\eta(\xi)$
where $\xi \in \left(0, C\int_0^a \frac{dt}{A^2(t)}\right)$
$\eta'(0) = 0$
$\eta'(\xi(a)) + \left(\frac{A'(a)A(a)}{C} + i\lambda\frac{A^2(a)}{C}\right)\eta(\xi(a)) = 0$
$\int_0^{\xi(a)} \frac{\rho(x)A^4(x)}{C}\left|\eta(\xi)\right| d\xi = 1.$ $(x = x(\xi))$

since $\xi'(x) = \frac{d\xi}{dx} = \frac{C}{A^2}$ then $dx = \frac{A^2}{C} d\xi$
If we take $A(x) = \left(\frac{C_1}{\rho(x)}\right)^{\frac{1}{4}}$ then $\rho(x)A^4(x) = C_1$
$\rightarrow \frac{\rho(x)A^4(x)}{C^2} = \frac{C_1}{C^2}$ where C_1 is constant
Introducing the new constants ρ and C_0 so that $\frac{C_1}{C^2} = \frac{\rho}{C_0^2}$ we obtain $\frac{\rho(x)A^4(x)}{C^2} = \frac{\rho}{C_0^2}$
$A^4(x)\rho(x)C_0^2 = \rho C^2 \rightarrow A^4(x) = \frac{\rho C^2}{\rho(x)C_0^2}$
$\rightarrow A(x) = \left(\frac{\rho C^2}{\rho(x)C_0^2}\right)^{\frac{1}{4}}$ (since $\rho(x) \in C^2_{[0,a]}$,then $A''(x)$ exist and is continuous)
With this choice of $A(x)$ we obtain: $-\eta''(\xi) + Q(\xi)\eta(\xi) = \lambda^2 \frac{\rho}{C_0^2}\eta(\xi(x)), \xi \in \left(0, C\int_0^a \frac{dt}{A^2(t)}\right)$ $\eta'(0) = 0$

$\eta'(\xi(a)) + \left(H_0 + ih_0.\frac{\lambda}{C_0}\right)\eta(\xi(a)) = 0$

Where $H_0 = \frac{A'(a)A(a)}{C}$, $h_0 = \frac{A^2(a)C_0}{C}$, $Q(\xi) = \left(\frac{q(x)A(x)-A''(x)}{C^2}\right)A^3(x)$
$\int_0^a \frac{\rho(x)A^4(x)}{C}.\frac{C}{C}\left|\eta(\xi(x))\right|^2 d\xi = 1$
If we put $\rho = \frac{\rho(a)}{h^2}$, and introduce the notation $\mu = \frac{\lambda}{C_0}$, $\xi(a) = a$, then we arrive to the problem:

$$-\eta''(\xi) + Q(\xi)\eta(\xi) = \mu^2\rho\eta(\xi), \qquad \xi \in (0, a), \tag{8}$$

$$\eta'(0) = 0, \eta'(\xi(a)) + (H_0 + ih_0\mu.)\eta(\xi(a)) = 0, \tag{9}$$

$$\int_0^a \frac{C\rho}{C^2}\left|\eta(\xi(x))\right|^2 d\xi = 1, \text{ where } \rho = \frac{\rho(\mathrm{a})}{\mathrm{h}^2} \tag{10}$$

Thus we come to the problem of the same type as (1)-(3), while $\rho(x) = \rho$.
The eigenvalues of (8)-(10) $\mu_n = \frac{\lambda_n}{C_0}(n = 1, 2,)$ where λ_n the eigenvalues of the problem (1)-(3). eigenfunctions of problem (1)-(3) and (8)-(10) are related by $y_n(x) = A(x).\eta_n(\xi(x))$ and clearly $\|y_n(x)\|_c = O(\|\eta_n(\xi)\|_c)$ and $\|\eta_n(\xi)\|_c = O(\|y_n(x)\|_c)$. It is also obvious that the replacement of condition (10) on $\int_0^a |\eta(\xi)|^2 d\xi = 1$ will only lead to a multiplication of the eigenfunctions by a constant. In view of this, in the case of smooth coefficients is sufficient to study the problem with $\rho(x) \equiv \rho \equiv constant$, that in the further and assume.

Furthermore, we assume that $h = 1$.

We introduce the notation:
$\varphi(x, \lambda)$ -solution of (1) with initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = 0$; $\varphi_0(x, \lambda)$ -solution of equation (1) with $q(x) \equiv q$ and initial conditions $\varphi_0(0, \lambda) = 1$,
${\varphi_0}'(0, \lambda) = 0$. thus $\varphi(x, \lambda)$- is asolution of (1) with any of the class factor $q(x)$, and
$\varphi_0(x, \lambda)$-the solution is more simple equation with constant coefficient $q(x) \equiv q$.
Obviously, the eigenvalues of (1)-(3) are precisely those λ , that satisfy the condition $\varphi'(a, \lambda) + i\lambda\varphi(a, \lambda) = 0$, and corresponding λ to these function are eigenfunctions $\varphi(x, \lambda)$ of the problem (1)-(3). since in the case of a constant coefficient $q(x) \equiv q$ problem (1)-(3) is easier to learn , we try to derive some relations between $\varphi(x, \lambda)$ and $\varphi_0(x, \lambda)$.
Since $\varphi(x, \lambda)$ the solution of (1), then we have

$$q(x)\varphi(x, \lambda) = \varphi''(x, \lambda) + \lambda^2\rho(x)\varphi(x, \lambda)$$

Replacing x by τ, we multiply this equation by $\varphi_0(x - \tau, \lambda)$ and integrate from 0 to x up $d\tau$, we get:

$$\int_0^x \varphi_0(x-\tau, \lambda)q(\tau)\varphi(\tau, \lambda)\,d\tau = \varphi_0(x-\tau, \lambda)\,\varphi'(\tau, \lambda)\Big|_0^x + {\varphi_0}'(x-\tau, \lambda)\,\varphi(\tau, \lambda)\Big|_0^x$$
$$+ \int_0^x {\varphi_0}''(x-\tau, \lambda)\,\varphi(\tau, \lambda)d\tau + \int_0^x \varphi_0(x-\tau, \lambda)\,.\lambda^2.\rho.\varphi(\tau, \lambda)\,d\tau$$

Or using the initial conditions and substituting for ${\varphi_0}''(x - \tau, \lambda)$ the value $q.\varphi_0(x - \tau, \lambda) - \lambda^2.\rho.\varphi_0(x - \tau, \lambda)$ out of the equation (1), we get:

$$\int_0^x \varphi_0(x-\tau, \lambda)q(\tau)\varphi(\tau, \lambda)\,d\tau = -\varphi_0(x, \lambda) + \varphi(x, \lambda) + \int_0^x q.\varphi_0(x-\tau, \lambda)\,.\varphi(\tau, \lambda)\,d\tau$$

Or

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x [q(\tau) - q].\varphi_0(x-\tau, \lambda)\,\varphi(\tau, \lambda)\,d\tau, \qquad (11)$$

Hence, substituting in the right side instead of $\varphi(\tau,\lambda)$ its value from the formula (11) and continuity this process indefinitely, we obtain:

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x [q(\tau_1)-q].\varphi_0(x-\tau_1,\lambda).\varphi_0(\tau_1,\lambda)\,\mathrm{d}\tau_1$$

$$+\sum_{i=2}^{\infty}\int_0^x [q(\tau_1)-q].\varphi_0(x-\tau_1,\lambda)\int_0^{\tau_1}\dots\int_0^{\tau_{i-1}}[q(\tau_i)-q].\varphi_0(\tau_{i-1}$$

$$-\tau_i,\lambda)\,\varphi(\tau_i,\lambda)\,\mathrm{d}\tau_i\mathrm{d}\tau_{i-1}\dots\mathrm{d}\tau_1 \tag{12}$$

We now estimate$|\varphi_0(x,\lambda)|$, using the explicit expression for $\varphi_0(x,\lambda)$. Since $\varphi_0(x,\lambda)$ solution of (1) with constant $q(x)\equiv q$ and initial condition ${\varphi_0}'(x,\lambda)$ $=0$, then the general form
$y(x)=C_1e^{i\sqrt{\lambda^2\rho-q}x}+C_2e^{-i\sqrt{\lambda^2\rho-q}\,x}$ of solutions ,we find that
$\varphi_0(0,\lambda)=1$ then $C_1+C_2=1$

$$y'(x)=iC_1\sqrt{\lambda^2\rho-q}e^{i\sqrt{\lambda^2\rho-q}x}-iC_2\sqrt{\lambda^2\rho-q}e^{-i\sqrt{\lambda^2\rho-q}\,x}$$

and ${\varphi_0}'(0,\lambda)=0$ then we have

$$iC_1\sqrt{\lambda^2\rho-q}-iC_2\sqrt{\lambda^2\rho-q}=0\rightarrow C_1=C_2$$

And since we have $C_1+C_2=1$

$$C_1=\frac{1}{2}$$

$$\varphi_0(x,\lambda)=C_1e^{i\sqrt{\lambda^2\rho-q}x}-C_1e^{-i\sqrt{\lambda^2\rho-q}\,x}$$

$$\rightarrow\varphi_0(x,\lambda)=C_1(e^{i\sqrt{\lambda^2\rho-q}x}-e^{-i\sqrt{\lambda^2\rho-q}\,x})$$

$$\rightarrow\varphi_0(x,\lambda)=\frac{1}{2}(e^{i\sqrt{\lambda^2\rho-q}x}-e^{-i\sqrt{\lambda^2\rho-q}\,x})$$

Using now the notation $\lambda=\delta+i\sigma$, we obtain:

$$\lambda^2\rho-q=(\delta+i\sigma)^2\rho-q=\left(\delta^2-\sigma^2+2i\delta\sigma\right)\rho-q=(\delta^2-\sigma^2)\rho-q$$

$$+2i\delta\sigma(\delta^2-\sigma^2)\rho-q+2i\delta\sigma\rho=(\delta_1+i\sigma_1)^2={\delta_1}^2-{\sigma_1}^2+2i\delta_1\sigma_1$$

${\delta_1}^2-{\sigma_1}^2=(\delta^2-\sigma^2)\rho-q$ and $2\delta_1\sigma_1=2\delta\sigma\rho\rightarrow\sigma=\frac{\delta_1\sigma_1}{\delta\sigma}$ & $\sigma_1=\frac{\delta\sigma\rho}{\delta_1}$

$${\delta_1}^2-{\sigma_1}^2=\left(-\delta^2-\frac{{\delta_1}^2{\sigma_1}^2}{\delta^2\rho^2}\right)\rho-q=\left(\frac{\delta^4\rho^2-{\delta_1}^2{\sigma_1}^2}{\delta^2\rho^2}\right)\rho-q$$

$${\delta_1}^2=\left(\frac{\delta^4\rho^2-{\delta_1}^2{\sigma_1}^2}{\delta^2\rho^2}\right)\rho-q+{\sigma_1}^2=\left(\frac{\delta^4\rho^2-{\delta_1}^2\frac{\delta^2\sigma^2\rho^2}{{\delta_1}^2}}{\delta^2\rho^2}\right)\rho-q+\frac{\delta^2\sigma^2\rho^2}{{\delta_1}^2}$$

$$=\left(\frac{\delta^4\rho^2-\delta^2\sigma^2\rho^2}{\delta^2\rho^2}\right)\rho-q+\frac{\delta^2\sigma^2\rho^2}{{\delta_1}^2}$$

$$=\left(\frac{\delta^2\rho^2\left(\delta^2-\sigma^2\right)}{\delta^2\rho^2}\right)\rho-q+\frac{\delta^2\sigma^2\rho^2}{{\delta_1}^2}$$

$$=\left(\left(\delta^2-\sigma^2\right)\rho-q\right)+\frac{\delta^2\sigma^2\rho^2}{{\delta_1}^2}$$

$${\delta_1}^4-\left(\left(\delta^2-\sigma^2\right)\rho-q\right){\delta_1}^2-\delta^2\sigma^2\rho^2=0$$

$${\delta_1}^2=\frac{\left(\left(\delta^2-\sigma^2\right)\rho-q\right)\mp\sqrt{\left[\left(\left(\delta^2-\sigma^2\right)\rho-q\right)\right]^2+4\delta^2\sigma^2\rho^2}}{2}$$

$$\rightarrow\delta_1=\mp\sqrt{\frac{\left(\delta^2-\sigma^2\right)\rho-q+\sqrt{\left[\left(\delta^2-\sigma^2\right)\rho-q\right]^2+4\delta^2\sigma^2\rho^2}}{2}}$$

And $\sigma_1=\frac{\delta\sigma\rho}{\delta_1}$

$$\rightarrow\sigma_1=\frac{\delta\sigma\rho}{\sqrt{\frac{\left(\delta^2-\sigma^2\right)\rho-q+\sqrt{\left[\left(\delta^2-\sigma^2\right)\rho-q\right]^2+4\delta^2\sigma^2\rho^2}}{2}}}$$

$$\sigma_1=\frac{\sqrt{2}\;\delta\sigma\rho}{\left(\delta^2-\sigma^2\right)\rho-q+\sqrt{\left[\left(\delta^2-\sigma^2\right)\rho-q\right]^2+4\delta^2\sigma^2\rho^2}}$$

$$= \sqrt{\frac{2\rho^2\delta^2\sigma^2}{\left(\delta^2 - \sigma^2\right)\rho - q + \sqrt{\left[\left(\delta^2 - \sigma^2\right)\rho - q\right]^2 + 4\delta^2\sigma^2\rho^2}}}$$

We transform $\varphi_0(x,\lambda)$, separating real and imaginary parts of each factor after simple transformation, we obtain:

$$\varphi_0(x,\lambda) = \frac{1}{2}\left(e^{i(\delta_1 + i\sigma_1)x} - e^{-i(\delta_1 + i\sigma_1)x}\right)$$

$$= \frac{1}{2}\left(e^{(-\sigma_1 + i\delta_1)x} - e^{(\sigma_1 - i\delta_1)x}\right)$$

$$= \frac{1}{2}\left(e^{-\sigma_1 x}\left(cos\delta_1 x + isin\delta_1 x\right) - e^{\sigma_1 x}\left(cos\delta_1 x - isin\delta_1 x\right)\right)$$

$$= \frac{1}{2}\left(e^{-\sigma_1 x}cos\delta_1 x - e^{\sigma_1 x}cos\delta_1 x + ie^{-\sigma_1 x}sin\delta_1 x + ie^{\sigma_1 x}sin\delta_1 x\right)$$

$$= \frac{1}{2}\left(\left(e^{-\sigma_1 x} - e^{\sigma_1 x}\right)cos\delta_1 x + i\left(e^{-\sigma_1 x} + e^{\sigma_1 x}\right)sin\delta_1 x\right)$$

$$= \left(\frac{e^{-\sigma_1 x} - e^{\sigma_1 x}}{2}cos\delta_1 x + i\left(\frac{e^{-\sigma_1 x} + e^{\sigma_1 x}}{2}\right)sin\delta_1 x\right)$$

$$= \left(-\sinh\sigma_1\ xcos\delta_1 x + i\cosh\sigma_1\ xsin\delta_1 x\right)$$

Now we find $|\varphi_0(x,\lambda)|$ using the resulting equation

$$|\varphi_0(x,\lambda)| = |-\sinh\sigma_1\ xcos\delta_1 x + i\cosh\sigma_1 xsin\delta_1 x\ |$$

$$= \sqrt{sinh^2\sigma_1 xcos^2\delta_1 x\ + cosh^2\sigma_1 xsin^2\delta_1 x}$$

$$= \quad \sqrt{sinh^2\sigma_1 xcos^2\delta_1 x\ + cosh^2\sigma_1 xsin^2\delta_1 x}$$

$$= \quad \sqrt{sinh^2\sigma_1 x\ + sin^2\delta_1 x}$$

We estimate

$$sh^2\sigma_1 x = sh^2 \frac{\sqrt{2}\ \rho\delta\sigma x}{\sqrt{\left(\delta^2-\sigma^2\right)\rho - q + \sqrt{\left[\left(\delta^2-\sigma^2\right)\rho - q\right]^2 + 4\delta^2\sigma^2\rho^2}}}$$

$$= sh^2 \frac{\sqrt{2}\ \rho\delta\sigma x}{\sqrt{\frac{(\delta^2-\sigma^2)\rho-q}{|(\delta^2-\sigma^2)\rho-q|} + \sqrt{\left[\left(\delta^2-\sigma^2\right)\rho - q\right]^2\left(1+\frac{4\delta^2\sigma^2\rho^2}{((\delta^2-\sigma^2)\rho-q)^2}\right)}}}$$

$$= sh^2 \frac{\sqrt{2}\ \rho\delta\sigma x}{\sqrt{\left|\left(\delta^2-\sigma^2\right)\rho - q\right|}\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2}\ \rho\delta\sigma x}{\sqrt{\delta^2\left|\left(1-\left(\frac{\sigma}{\delta}\right)^2\right)\rho-\frac{q}{\delta^2}\right|}\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2}\ \rho\delta\sigma x}{\delta\sqrt{\left|\left(1-\left(\frac{\sigma}{\delta}\right)^2\right)\rho-\frac{q}{\delta^2}\right|}\qquad\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2}\ \rho\sigma x}{\sqrt{\left|\left(1-\left(\frac{\sigma}{\delta}\right)^2\right)\rho-\frac{q}{\delta^2}\right|}\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2}\quad\sqrt{\rho}\quad\sqrt{\rho}\sigma x}{\sqrt{\rho}\sqrt{\left|\left(1-\left(\frac{\sigma}{\delta}\right)^2\right)-\frac{q}{\rho\delta^2}\right|}\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2}\quad\sqrt{\rho}\quad\sigma x}{\sqrt{\left(1-\left(\frac{\sigma^2}{\delta^2}+\frac{q}{\rho\delta^2}\right)\right)}\sqrt{\sqrt{1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2}+1}}$$

$$= sh^2 \frac{\sqrt{2} \quad \sqrt{\rho} \quad \sigma x}{\left(1-\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)\right)^{\frac{1}{2}}\left(1+\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\rho-q}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

$$= sh^2\left(\sqrt{2} \quad \sqrt{\rho} \quad \sigma x\left(1-\frac{1}{2}\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)+\frac{3}{8}\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)^2-\dots\right)\left(1-\frac{1}{2}\right.\right.$$

$$\left.\left.\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\,\rho-q}\right)^2\right)^{\frac{1}{2}}+\frac{3}{8}\left(\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\,\rho-q}\right)^2\right)^{\frac{1}{2}}\right)^2-\dots\right)\right)$$

$$= sh^2\left(\sqrt{2}\,\sqrt{\rho}\,\sigma x\left(1-\frac{1}{2}\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\,\rho-q}\right)^2\right)^{\frac{1}{2}}+\dots-\frac{1}{2}\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)\right.\right.$$

$$\left.\left.+\frac{1}{4}\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\,\rho-q}\right)^2\right)^{\frac{1}{2}}+\dots\right)\right)$$

$$\leq sh^2\left[\sqrt{\rho}\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma x\right] \leq sh^2\left[\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma\right]$$

Where $\zeta > 0$ (since shx is monotonically increasing) where

$$\zeta\left(\frac{\sigma}{\delta}\right)^2 = \left(-\frac{1}{2}\left(1+\left(\frac{2\rho\delta\sigma}{(\delta^2-\sigma^2)\,\rho-q}\right)^2\right)^{\frac{1}{2}}+\dots-\frac{1}{2}\left(\frac{\rho\sigma^2+q}{\rho\delta^2}\right)+\dots\right)$$

since $sh^2\left[\sqrt{\rho}\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma x\right] = \left[\frac{e^{\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma} - -e^{-\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma}}{2}\right]^2$

$$= \frac{e^{2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma} - 2 + e^{-2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma}}{4}$$

$$= \frac{e^{2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma}}{4}\left[1-2e^{-2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma}+e^{-4\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)\sigma}\right]$$

$$= \frac{e^{2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma|}}{4}\left[1-e^{-2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma|}\right]^2$$

$$= \frac{e^{2\sqrt{\rho}\,a\;|\sigma|}}{4} e^{2\sqrt{\rho}\,a\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|} \left[1-\left(1-2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma| + \frac{\left(-2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma|\right)^2}{2!}+\dots\right)\right]^2$$

$$= \frac{e^{2\sqrt{\rho}\,a\;|\sigma|}}{4} e^{2\sqrt{\rho}\,a\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|} \left[2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma|\,(1- \frac{2\sqrt{\rho}\,a\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2\right)|\sigma|}{2!}\dots\right)\right]^2$$

Then we obtain $sh^2\sigma_1 x < \frac{e^{2\sqrt{\rho}\,a\;|\sigma|}}{4}\left[1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|\right]$. For sufficiently large δ (more precisely, if $\left(\frac{\sigma}{\delta}\right)^2|\sigma|$ a little). And since, $|\varphi_0(x,\lambda)| = \sqrt{sinh^2\sigma_1 x + sin^2\delta_1 x}$. From the inequality $sin^2\delta_1 x \le 1$ and the above estimates imply, that

$$|\varphi_0(x,\lambda)| < \sqrt{\frac{e^{2\sqrt{\rho}\,x\;|\sigma|}}{4}\left[1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|\right]+1}$$

$$= \sqrt{\frac{e^{2\sqrt{\rho}\,x\;|\sigma|\left[1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|\right]+4}}{4}}$$

$$= \frac{\sqrt{e^{2\sqrt{\rho}\,x\;|\sigma|}\left[1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+4e^{-2\sqrt{\rho}\,x\;|\sigma|}\right]}}{2}$$

$$= \frac{e^{\sqrt{\rho}\,x\;|\sigma|}\sqrt{1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+4e^{-2\sqrt{\rho}\,x\;|\sigma|}}}{2}$$

$$< \frac{e^{\sqrt{\rho}\,x\;|\sigma|}}{2}\left(1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+4e^{-2\sqrt{\rho}\,x\;|\sigma|}\right)^2$$

$$= \frac{e^{\sqrt{\rho}\,x\;|\sigma|}}{2}\left(1+2\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+\left(\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|\right)^2+8e^{-2\sqrt{\rho}\,x\;|\sigma|}+8e^{-2\sqrt{\rho}\,x\;|\sigma|}\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+16e^{-4\sqrt{\rho}\,x\;|\sigma|}\right)$$

$$< \frac{e^{\sqrt{\rho}\,x\,|\sigma|}}{2}\left[1+8e^{-2\sqrt{\rho}\,x\,|\sigma|}+\zeta\left(\frac{\sigma}{\delta}\right)^{2}|\sigma|+16e^{-4\sqrt{\rho}\,x\,|\sigma|}\right] \tag{13}$$

Transposing in (12) $\varphi_0(x,\lambda)$ in to the left side, using the estimate $|q(x)-q| < Q_0$ and the estimate (13) on the right side of this equation, we arrive to the inequality $|\varphi(x,\lambda)-\varphi_0(x,\lambda)| < \sum_{i=1}^{\infty} {Q_0}^{i}\,.\zeta^{i+1}.\frac{e^{\sqrt{\rho}\,x\,\sigma}}{2}.\frac{x^i}{i!}$

$$|\varphi(x,\lambda)-\varphi_0(x,\lambda)| < C_0.\frac{e^{\sqrt{\rho}\,a\sigma}}{2} \quad (C_0>0 \textit{ new constant}) \tag{14}$$

Since $|\varphi_0(x,\lambda)| = \sqrt{sinh^2\sigma_1 x\ + sin^2\delta_1 x} \geq \sqrt{sinh^2\sigma_1 x}$, *then for* $x \geq a_0 > 0$ Exactly the same way was as the estimates were obtained (13) we obtain lower bounds for $|\varphi_0(x,\lambda)|$ if $|\sigma| \geq \sigma_0 > 0$.

$$= \sqrt{\frac{e^{2\sqrt{\rho}\,x\,|\sigma|\left[1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|\right]}}{4}}$$

$$= \frac{e^{\sqrt{\rho}\,x\,|\sigma|}\sqrt{1+\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|}}{2}$$

$$= \frac{e^{\sqrt{\rho}\,x\,|\sigma|}}{2}\left(1-\frac{1}{2}\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+\ldots\right)$$

$$= \frac{e^{\sqrt{\rho}\,x\,|\sigma|}}{2}\left(1-\frac{1}{2}\zeta\left(\frac{\sigma}{\delta}\right)^2|\sigma|+\ldots\right) > C_1.\frac{e^{\sqrt{\rho}\,x\,|\sigma|}}{2}$$

Then we have

$$|\varphi_0(x,\lambda)| > C_1.\frac{e^{\sqrt{\rho}\,x\,\sigma}}{2} \quad (a_0 \leq x \leq a) \tag{15}$$

If $|\delta| \gg |\sigma|$ $\left(\left|\frac{\delta}{\sigma}\right| \textit{ sufficiently large}\right)$, *then we have the following*

Remark:

For all sufficiently large $|\lambda|$ we have the inequalities:

$$B_0.\frac{\|\varphi_0(x,\lambda)\|_c}{\left(\int_0^1 \rho\,|\varphi_0^2(x,\lambda)|\,dx\right)^{\frac{1}{2}}} < \frac{\|\varphi(x,\lambda)\|_c}{\left(\int_0^1 \rho\,|\varphi^2(x,\lambda)|\,dx\right)^{\frac{1}{2}}} < B_1.\frac{\|\varphi_0(x,\lambda)\|_c}{\left(\int_0^1 \rho\,|\varphi_0^2(x,\lambda)|\,dx\right)^{\frac{1}{2}}}$$

Where $0< B_0 < B_1 < \infty$ *constant, independent on* λ. This follows directly from (8),(9) and (10). Namely, substituting in the inequality of the statement instead of $\varphi(x,\lambda)$ the expression $\varphi_0(x,\lambda)+\triangle(x,\lambda)$ and using (10) for $\triangle(x,\lambda)$ and (9) and (11) for $\varphi_0(x,\lambda)$ we obtain proof. since we already have the eigenfunctions of the problem (1)-(3) with constant coefficients and asymptotic behavior for eigenvalues of (1)-(3) in the case $h = 1$, is known [13] and in all case ($\rho = 1$ and $\rho \neq 1$) the eigenvalues do indeed satisfy the relation $|\delta| \gg |\sigma|$, the asymptotic behavior of eigenfunctions for $q(x) \equiv$ constant and arbitrary $q(x) \in C_{[0,a]}$ is the same.

4. Conclusion

In the present work , we showed the asymptotic of eigen values in both cases regular and irregular and also proved the estimations of normalized eigenfunctions the problem (1)-(3) are uniformly bounded with the coefficients are smooth.

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