ON GENERALIZED PROBABILISTIC METRIC SPACES

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Abstract. In the present paper we study some generalized probabilistic metric spaces. Relationships with another deterministic and probabilistic metric structures are analyzed. A contraction condition for mappings with values into such a generalized probabilistic metric space is given. Fixed point results are proved.

Key words and phrases: generalized probabilistic metric space, probabilistic contraction, fixed point.

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1. Introduction

In [11] K. Menger proposed a probabilistic concept of distance by replacing the number $d(p, q)$, the distance between points p, q by a distribution function $F_{p,q}$. This idea led to a large development of probabilistic analysis [2], [8] [12]. The idea of a n-dimensional metric has also appeared first in K. Menger's papers [10]. Three decades later S. Gähler formulated an appropriate system of axioms for a distance between three points and developed the theory of 2-metric spaces [5].
An enlargement of the concept of 2-metric space was given in [3], where a study of generalized metric spaces is developed.
Now, we recall some standard notions and notations. Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $I = [0, 1]$ the closed unit interval. A mapping $F : \mathbb{R} \to I$ is called a distribution function if it is non decreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$.
D_+ denotes the set of all distribution functions for that $F(0) = 0$. Let F, G be in D_+ , then we write $F \leq G$ if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. If $a \in \mathbb{R}_+$ then H_a will be the element of D_+, for which $H_a(t) = 0$ if

$t \le a$ and $H_a(t) = 1$ if $t > a$. It is obvious that $H_0 \ge F$, for all $F \in D_+$. The set D_+ will be endowed with the natural topology defined by the modified Lévy metric d_L [10]. The modified Levy metric d_L induces on D_+ the topology of weak convergence, and the following properties are verified :

(1) $F(t) > 1 - t$ if and only if $d_L(F, H_0) < t$.

(2) If $F \leqslant G$ then $d_L(G, H_0) \leqslant d_L(F, H_0)$.

(3) The metric space (D_+, d_L) is compact, and hence complete.

A t-norm T_1 is a two place function $T_1 : I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $T_1(a, 1) = a$, for all $a \in [0, 1]$. A triangle function τ_1 is a binary operation on D_+ which is commutative, associative and for which H_0 is the identity, that is, $\tau_1(F, H_{0)}) = F$, for every $F \in D_+$ [2],[12].

In [3] B. C. Dhage formulated the following system of axioms for a distance between three points and developed a theory of generalized metric spaces.

Definition 1.1. *Let X be a non empty set. A generalized metric space is a pair (X, d), where d is a mapping from $X \times X \times X$ into $\mathbb{R}_+$ and the following conditions are satisfied :*

(4) $d(x, y, z) = 0$ *if and only if* $x = y = z$.

(5) $d(x, y, z) = 0$ *if at least two of* x, y, z *are equal.*

(6) $d(x, y, z) = d(x, z, y) = d(y, z, x)$*, for every* x, y, z *in* X.

(7) $d(x, y, z) \leqslant d(x, y, u) + d(x, u, z) + d(u, y, z)$*, for every* x, y, z, u *in* X.

Geometrically, the 2-metric between three points defined in [5] measures the area of the triangle having as vertices these points, while the generalized metric defined in [3] measures the perimeter of the same triangle.

2. Generalized probabilistic metric spaces

Let T_1 be a t-norm and let τ be a triangle function. In the sequel we will use the functions $T : [0, 1]^3 \to [0, 1]$ given by $T(a, b, c) = T_1(T_1(a, b), c)$ and $\tau : [D_+]^3 \to D_+$ given by $\tau(F, G, H) = \tau_1(\tau_1(F, G), H)$, we name T a th-norm and τ a th-function.

They have appropriate properties for writing a triangle inequality in generalized probabilistic metric spaces. In [1] a generalized class of t-norms on $[0, 1]^3$ was defined , but they are, in fact, th-norms.

Definition 2.1. *A generalized probabilistic metric space is an ordered triple ($X, \mathcal{F}, \tau$), where X is a non empty set, $\mathcal{F}$ is a function defined on $X \times X \times X$ with values into D_+, τ is a th-function and the following conditions are satisfied:*

(8) $F_{x,y,z} = H_0$ *if and only if* $x = y = z$,
(9) $F_{x,y,z} = H_{x,z,y} = H_{y,z,x}$, *for every* x, y, z *in* X.
(10) $F_{x,y,z} \geqslant \tau(F_{x,y,u}, F_{x,u,z}, F_{u,y,z})$, *for every* x, y, z, u *in* X.

The inequality (10), named and tetrahedral inequality can be given by a th-norm T by :

(11) $F_{x,y,z}(t) \geqslant T(F_{x,y,u}((t_1), F_{x,u,z}(t_2), F_{u,y,z}(t_3))$, for every $t_1, t_2, t_3 \in \mathbb{R}_+$ such that $t_1 + t_2 + t_3 = t$. In this case $(X, \mathcal{F}, T)$ is called a generalized Menger metric space.

It is easy to check that every generalized metric space (X, d) can be made, in a natural way, a generalized Menger metric space by setting $F_{x,y,z}(t) = H_0(t - d(x, y, z)))(t)$, for every $x, y \in X$, $t \in \mathbb{R}_+$ and $T = Min$.

The relationship between the two class of generalized metric spaces is given by the following statement.

Proposition 2.2. *If* T *is a left continuous th-norm and* τ_T *is the th-function defined by* $\tau_T(F, G, H)(t) = \sup_{t_1+t_2+t_3<t} T(F(t_1), G(t_2), H(t_3))$, $t > 0$, *then* $(X, \mathcal{F}, \tau_T)$ *is a probabilistic D-metric space if and only* $(X, \mathcal{F}, T)$ *is a Menger D-metric space.*

Definition 2.3. *A sequence* $\{x_n\}$ *of points in a generalized probabilistic metric space* $(X, \mathcal{F}, \tau)$ *is said to be convergent to the point* $x \in X$ *if for each* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$F_{x_n,x_m,x}(t) > 1 - t,$$

for all $n, m \geqslant n_0$.

Definition 2.4. *We say that a sequence* $\{x_n\}$ *of probabilistic D-metric space* $(X, \mathcal{F}, tau)$ *is a Chauchy sequence if for each* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$F_{x_n,x_m,x_p}(t) > 1 - t,$$

for all $m, p > n \geqslant n_0$.

Definition 2.5. *A generalized probabilistic metric space* $(X, \mathcal{F}, T)$ *is said to be complete if every Cauchy sequence under the probabilistic metric* $\mathcal{F}$ *converges to a point* $x \in X$

Definition 2.6. *A self mapping* f *of a probabilistic D-metric space* $(X, \mathcal{F}, T)$ *is said to be continuous if* $fx_n \to fx$, *whenever* $x_n \to x$.

Proposition 2.7. *Let* $\{x_n\}$ *be a sequence of points in a generalized probabilistic metric space* $(X, \mathcal{F}, T,)$ T *be a continuous th-norm* T. *Then we have:*

(a) $x_n \to x$, *if and only if* $d_L(F_{x_n,x_m,x}, H_0) \to 0, (n, m \to \infty)$.
(b) $x_n \to x$ *if and only if* $F_{x_n,x_m,x}(t) \to H_0(t)$, *for all* $t > 0$.
(c) $\{x_n\}$ *is a Cauchy sequence if and only if* $d_L(F_{x_n,x_m,x_p}, H_0 \to 0$ $(n, m, p \to \infty)$.
(d) $\{x_n\}$ *is a Cauchy sequence if and only if* $F_{x_n,x_m,x_p}(t) \to H_0(t)$ *, for all* $t > 0$.

Example 2.8. *Let* $(L, \|.\|)$ *be a separable Banach space and let* $(L, \mathcal{B})$ *be the measurable space, where* $\mathcal{B}$ *is the* σ*-algebra of Borel subsets of the separable Banach space* $(L, ||.||)$. *We denote by* X *the linear space of all random variables defined on a probability measure space* $(\Omega, \mathcal{K}, P)$ *with values in* $(L, \mathcal{B})$.
For all $x, y, z \in X$, $t \in \mathcal{R}$, *and* $t > 0$ *we define the mapping* $\mathcal{F} : X^3 \to D_+$ *given by* $\mathcal{F}(x, y, z) = F_{x,y,z}(t)$, *where*

$$F_{x,y,z}(t) = P(\{\omega \in \Omega : ||x(\omega) - y(\omega)|| + ||x(\omega) - y(\omega)|| + ||y(\omega) - z(\omega)|| < t\}).$$

The triple $(X, \mathcal{F}, T_m)$ *becomes a generalized probabilistic metric space.*
The following theorem gives a relationship between a generalized probabilistic metric space and a probabilistic metric space.
Theorem 2.9. *Let* $(X, \mathcal{F}, T)$ *be a Menger space which has at least three points and let* $\mathcal{F} : X^3 \to D_+$ *a mapping given by*

$$\mathcal{F}(x, y, z) = F_{x,y,z}(t) = Min\{F_{x,y}(t), F_{y,z}(t), F_{z,x}(t)\},$$

then the triple $(X, \mathcal{F}, Min)$ *is a generalized Menger space.*
Now, we show that some generalized Menger spaces $(X, \mathcal{F}, T)$ can be endowed with a generalized metric that induces the same convergence with the generalized probabilistic metric $\mathcal{F}$.
Theorem 2.10. *Let* $(X, \mathcal{F}, T)$ *be a generalized Menger space under a continuous th-norm* T *such that* $T \geqslant T_m$ *and let consider the mapping* $d : X^3 \to \mathcal{R}$ *defined by*

$$d(x, y, z) = sup\{\varepsilon \in [0, 1) : F_{x,y,z}(\varepsilon) \leqslant 1 - \varepsilon\}.$$

Then we have :

(a) $d(x, y, z) < t$ *if and only if* $F_{x,y,z}(t) > 1 - t$.
(b) (X, d) *is a generalized metric space.*
(c) *The convergence under the generalized probabilistic metric* $\mathcal{F}$ *is equivalent with convergence under the generalized metric* d.

3. A fixed point theorem in a generalized probabilistic metric space

A first type of contraction conditions in probabilistic metric spaces was first given [13], fixed point theorems were also obtained. Later a second type of contraction mappings was introduced in [9]. Since, many results were obtained [2],[8],[10]. In what sequel we study a type of contraction in generalized probabilistic metric spaces and we give a fixed point theorem.
Let us consider a function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ such that the following conditions are satisfied :
(a_1) φ is nondecreasing and right continuous;
(a_2) $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, for all $t > 0$;
(a_3) there is $t > 0$ such that $\varphi(t) > 1$, ;
It is easy to see that under these conditions $\varphi(t) < t$. We denote by Φ the set of all functions which satisfy the conditions (a_1), (a_2), and a_3. The family of functions $\varphi_{k,n}(t) = k^n t$, $k \in (0,1)$ and $n \in \mathbf{N}$ is into the set Φ. Now, let φ be in Φ.
Definition 3.1. *Let $(X, \mathcal{F}, T)$ be a generalized Menger space under a continuous th-norm T. A mapping $f : X \to X$ which satisfies the following condition :*

$$(c) \quad \textit{If} \quad t > 0 \quad \textit{and} \quad F_{x,y,z}(t) > 1 - t, \quad \textit{then} \quad F_{fx,fy,fz}(\varphi(t)) > 1 - \varphi(t)$$

is called $\varphi-$contraction.
The definition seems to be natural because we mean in a particular case that, under a probability measure, the perimeter of the triangle whose vertices are fx, fy, fz is less than the perimeter of the triangle whose vertices are x, y, z.
Theorem 3.2. *Let $(X, \mathcal{F}, T)$ be a generalized Menger space under a continuous th-norm T. Then a φ-contraction $f : X \to X$ has a unique fixed point which is the limit of the sequence $\{x_n\}$ defined by $x_0 \in X$ and $x_{n+1} = fx_n$, $n \geqslant 0$.*
By the above theorems fixed point results can be translated between probabilistic and deterministic generalized metric spaces.

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