UNIVALENCE FOR INTEGRAL OPERATORS

by
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Abstract: We study in this paper two integral operators and we determine the conditions for univalence for these integral operators.

Let A be the class of the functions f, which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f^{'}(0) - 1 = 0$ and denote by S the class of univalent functions.

Teorema 1 [4] If $f \in S$, $\alpha \in C$ and $|\alpha| \le 1/4$ then the function:

$$G_\alpha(z) = \int_0^z [f'(t)]^\alpha \, dt$$

is univalent in U.

Teorema 2 [3] If the function $g \in S, \alpha \in C, |\alpha| \le 1/4n$, then the function defined by

$$G_{\alpha,n}(z) = \int_0^z [f'(t^n)]^\alpha \, dt$$

is univalent in U for $n \in N^*$.

Teorema 3 [2]

Let $\alpha, a \in C, \operatorname{Re}\alpha > 0$ and $f \in A$.

If

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \quad (\forall) z \in U$$

then $(\forall)\beta \in C,\ \operatorname{Re}\beta \ge \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) \right]^{1/\beta}$$

is univalent.

Teorema 4 [1] If the function g is holomorphic in U and $|g(z)| < 1$ in U then $(\forall)\xi \in U$ and $z \in U$ the following inequalities hold:

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right|, \qquad (1)$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

the equalities hold in case $g(z) = \frac{\varepsilon(z + t)}{1 + \overline{t}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 5 [1]

For *z = 0*, the inequalities (1) are the form:

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq \xi$$

and

$$|g(\xi)| = \frac{|\xi| + |g(0)|}{1 + |g(0)| \cdot |\xi|}.$$

Considering g(0) = a and $\xi = z \Rightarrow |g(z)| \leq \frac{|z| + |a|}{1 + |a| \cdot |z|} \quad (\forall)z \in U$.

Lemma 6 (Schwartz)

If the function g is holomorphic in U, g(0) = 0 and $|g(z)| \leq 1$ $(\forall)z \in U$, then:

$$|g(z)| \leq |z| \quad (\forall)z \in U \text{ and } |g'(0)| \leq 1$$

the equalities hold in case g(z) = εz, where ε =1.

Theorem 7. Let α, β, γ, δ, complex number with the properties *Reδ = a >0, f,g*$\in A$ $f = z + a_2 z^2 + \ldots$, $g = z + b_2 z^2 + \ldots$,

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{n}, (\forall)z \in U, \qquad (2)$$
$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{1}{n}, (\forall)z \in U,$$

$$|\alpha\beta| \leq \frac{n + 2a}{2} \cdot \left(\frac{n + 2a}{n}\right)^{\frac{n}{2a}} \qquad (3)$$

and

$$\frac{1}{|\alpha|}+\frac{1}{|\beta|}<1. \qquad (4)$$

Then $(\forall)\gamma \in C, \operatorname{Re}\gamma \geq a$ the function:

$$D_{\alpha,\beta,\gamma,n}(z)=\left\{\gamma\int_0^z t^{\gamma-1}\cdot\left[f'(t^n)\right]^{\alpha}\cdot\left[g'(t^n)\right]^{\beta}dt\right\}^{1/\gamma}$$

is univalent $(\forall)n \in N^* \setminus \{1\}$.

Proof: We consider the function:

$$h(z)=\int_0^z [f'(t^n)]^{\alpha}[g'(t^n)]^{\beta}\,dt \qquad (5)$$

The function

$$p(z)=\frac{1}{|\alpha\beta|}\cdot\frac{h''(z)}{h'(z)} \qquad (6)$$

where $|\alpha\beta|$ satisfies the conditions (3).

We obtain:

$$p(z)=\frac{1}{|\alpha\beta|}\cdot\frac{h''(z)}{h'(z)}=\frac{1}{|\alpha\beta|}\cdot\frac{\left\{[f'(t^n)]^{\alpha}\cdot[g'(t^n)]^{\beta}\right\}'}{[f'(t^n)]^{\alpha}\cdot[g'(t^n)]^{\beta}}=$$

$$=\frac{1}{|\alpha\beta|}\cdot\frac{\alpha\cdot n\cdot z^{n-1}\cdot\left(f'(z^n)\right)^{\alpha-1}\cdot f''(z^n)\cdot\left[g'(z^n)\right]^{\beta}+\left[f'(z^n)\right]^{\alpha}\cdot\beta\cdot n\cdot z^{n-1}\left(g'(z^n)\right)^{\beta-1}\cdot g''(z^n)}{\left[f'(z^n)\right]^{\alpha}\cdot\left[g'(z^n)\right]^{\beta}}=$$

$$=\frac{\alpha}{|\alpha\beta|}\cdot\frac{\alpha\cdot n\cdot z^{n-1}\cdot f''(z^n)}{f'(z^n)}+\frac{\beta}{|\alpha\beta|}\cdot\frac{n\cdot z^{n-1}\cdot g''(z^n)}{g'(z^n)} \qquad (7)$$

We respect the conditions (2), (4) and (7) we have:

$$|p(z)| = \left| \frac{\alpha}{|\alpha\beta|} \cdot \frac{nz^{n-1} \cdot f''(z^n)}{f'(z^n)} + \frac{\beta}{|\alpha\beta|} \cdot \frac{nz^{n-1} \cdot g''(z^n)}{g'(z^n)} \right| \leq$$

$$\leq \left| \frac{\alpha}{|\alpha\beta|} \cdot \frac{nz^{n-1} \cdot f''(z^n)}{f'(z^n)} \right| + \left| \frac{\beta}{|\alpha\beta|} \cdot \frac{n \cdot z^{n-1} \cdot g''(z^n)}{g'(z^n)} \right| \leq$$

$$\leq \frac{|\alpha|}{|\alpha||\beta|} \cdot n \cdot \left| z^{n-1} \right| \cdot \left| \frac{f''(z^n)}{f'(z^n)} \right| + \frac{|\beta|}{|\alpha||\beta|} \cdot n \cdot \left| z^{n-1} \right| \cdot \left| \frac{g''(z^n)}{g'(z^n)} \right| \leq$$

$$\leq \frac{1}{|\beta|} \cdot n \cdot \frac{1}{n} + \frac{1}{|\alpha|} \cdot n \cdot \frac{1}{n} = \frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1 .$$

p (0) = 0

By Schwartz lemma we have:

$$\frac{1}{|\alpha\beta|} \cdot \left| \frac{h''(z)}{h'(z)} \right| \leq |z|^{n-1} \leq |z| \Leftrightarrow \left| \frac{h''(z)}{h'(z)} \right| \leq |\alpha\beta| \cdot \left| z^{n-1} \right| \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1-|z|^{2a}}{a} \right) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq |\alpha\beta| \cdot \left(\frac{1-|z|^{2a}}{a} \right) \cdot |z|^n \qquad (8)$$

Let's the function $Q:[0,1] \to R,\ \ Q(x) = \frac{1-x^{2a}}{a} \cdot x^n, x = |z|$.

We have

$$Q(x) \leq \frac{2}{n+2a} \cdot \left(\frac{n}{n+2a} \right)^{\frac{n}{2a}}, \quad (\forall) x \in [0,1].$$

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Respect the conditions (3) and (8) we obtain:

$$\left(\frac{1-|z|^{2a}}{a}\right)\cdot\left|\frac{z\cdot h''(z)}{h'(z)}\right|\le 1.$$

In this conditions applying the theorem 3 we obtain:

$$D_{\alpha,\beta,\gamma,n}(z)=\left\{\gamma\int_0^z t^{\gamma-1}\cdot\left[f'(t^n)\right]^{\alpha}\cdot\left[g'(t^n)\right]^{\beta}dt\right\}^{1/\gamma}$$

is univalent $(\forall)n\in N^*\setminus\{1\}$.

Corollary 8 [5]. Let $\alpha,\gamma\in C, \operatorname{Re}\alpha=a>0,\ g\in A$

If

$$\left|\frac{g''(z)}{g'(z)}\right|\le\frac{1}{n}\qquad(\forall)z\in U$$

and

$$|\gamma|\le\frac{n+2a}{2}\left(\frac{n+2a}{n}\right)^{n/2a}$$

then $(\forall)\beta\in C,\ \operatorname{Re}\beta\ge a,$ the function

$$G_{\beta,\gamma,n}(z)=\left\{\beta\int_0^z t^{\beta-1}\left[g'(t^n)\right]^{\gamma}\right\}^{1/\beta}$$

is univalent $(\forall)n\in N^*\setminus\{1\}$.

Theorem 9. Let α, β, γ, δ complex number, $\operatorname{Re}\delta=c>0$, and the functions $f,g\in A$ $f=z+a_2z^2+\dots,\ g=z+b_2z^2+\dots,$.

If f and g satisfy the conditions

$$\left|\frac{f''(z)}{f'(z)}\right|<1\qquad\text{and}\qquad\left|\frac{g''(z)}{g'(z)}\right|<1\quad(\forall)z\in U,\qquad(9)$$

and α with β the conditions:

$$\frac{1}{|\alpha|}+\frac{1}{|\beta|}<1\qquad(10)$$

and

$$|\alpha\beta| \le \frac{1}{\max\limits_{|z|\le 1}\left[\frac{1-|z|^{2c}}{c}\cdot|z|\cdot\frac{|z|+2\cdot\frac{|\alpha a_2+\beta b_2|}{|\alpha\beta|}}{1+2\cdot\frac{|\alpha a_2+\beta b_2|}{|\alpha\beta|}|z|}\right]} \qquad (11),$$

then $(\forall)\gamma \in C, \mathrm{Re}\,\gamma \ge c$ the function

$$F_{\alpha,\beta,\gamma}(z) = \left\{\gamma\cdot\int_0^z t^{\gamma-1}[f'(t)]^\alpha\cdot[g'(t)]^\beta\right\}^{1/\gamma}$$

is univalent.

Proof:

Let

$$h(z) = \int_0^z [f'(t)]^\alpha\cdot[g'(t)]^\beta dt\,.$$

Let

$$p(z) = \frac{1}{|\alpha\beta|}\cdot\frac{h''(z)}{h'(z)}\,.$$

Then:

$$p(z) = \frac{1}{|\alpha\beta|}\cdot\frac{\alpha(f'(z))^{\alpha-1}\cdot f''(z)\cdot(g'(z))^\beta}{(f'(z))^\alpha\cdot(g'(z))^\beta} + \frac{1}{|\alpha\beta|}\cdot\frac{(f'(z))^\alpha\cdot\beta\cdot(g'(z))^{\beta-1}\cdot g''(z)}{(f'(z))^\alpha\cdot(g'(z))^\beta} =$$

$$= \frac{\alpha}{|\alpha\beta|}\cdot\frac{f''(z)}{f'(z)} + \frac{\beta}{|\alpha\beta|}\cdot\frac{g''(z)}{g'(z)}\,.$$

The function p is olomorf , $(\forall)z \in U$.

We respect the conditions (9) and (10) obtain:

$$|p(z)| = \left|\frac{\alpha}{|\alpha\beta|}\cdot\frac{f''(z)}{f'(z)} + \frac{\beta}{|\alpha\beta|}\cdot\frac{g''(z)}{g'(z)}\right| \le \left|\frac{\alpha}{|\alpha\beta|}\cdot\frac{f''(z)}{f'(z)}\right| + \left|\frac{\beta}{|\alpha\beta|}\cdot\frac{g''(z)}{g'(z)}\right| = \frac{1}{|\alpha|}+\frac{1}{|\beta|} < 1$$

$(\forall)z \in U$.

$$|p(0)| = \left|\frac{\alpha}{|\alpha\beta|}\cdot\frac{f''(0)}{f'(0)} + \frac{\beta}{|\alpha\beta|}\cdot\frac{g''(0)}{g'(0)}\right| = \left|\frac{\alpha}{|\alpha\beta|}\cdot 2a_2 + \frac{\beta}{|\alpha\beta|}\cdot 2b_2\right| = 2\cdot\frac{|\alpha a_2+\beta b_2|}{|\alpha\beta|}$$

Respecting the remark (5) we have:

$$|p(z)| \le \frac{|z| + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|} \cdot |z|}, \quad (\forall) z \in U$$

$$\Leftrightarrow \frac{1}{|\alpha\beta|} \cdot \left| \frac{h''(z)}{h'(z)} \right| \le \frac{|z| + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2 \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|} |z|}, \ (\forall) z \in U \Leftrightarrow$$

$$\Leftrightarrow \frac{1 - |z|^{2c}}{c} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \le |\alpha\beta| \cdot \frac{1 - |z|^{2c}}{c} \cdot |z| \cdot \frac{|z| + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|} \cdot |z|}, \quad (\forall) z \in U .$$

The last inequality implies:

$$\frac{1 - |z|^{2c}}{c} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \le |\alpha\beta| \cdot \max_{|z| \le 1} \left[\frac{1 - |z|^{2c}}{c} \cdot |z| \cdot \frac{|z| + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2 \cdot \dfrac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|} \cdot |z|} \right]$$

The relation (11) and the last inequality implies:

$$\frac{1 - |z|^{2c}}{c} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \le 1, (\forall) z \in U$$

and by the theorem 3 we obtain

$$F_{\alpha,\beta,\gamma,}(z) = \left\{ \gamma \cdot \int_0^z t^{\gamma - 1} [f'(t)]^{\alpha} \cdot [g'(t)]^{\beta} \right\}^{1/\gamma}$$

is univalent.

Corollary 10 [5] Let $\alpha, \gamma \in C, \operatorname{Re}\alpha = b > 0$ and function $g \in A, g(z) = z + a_2 z^2 + ...$

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$$\left|\frac{g''(z)}{g'(z)}\right| < 1, \qquad (\forall) z \in U$$

and γ satisfy the condition:

$$|\gamma| \le \frac{1}{\max_{|z|\le 1}\left[\left(\frac{1-|z|^{2b}}{b}\cdot|z|\right)\cdot\left(\frac{|z|+2|a_2|}{1+2|a_2||z|}\right)\right]}$$

then for all $\beta \in C$, $\operatorname{Re}\beta \ge b$, the function

$$G_{\beta,\gamma}(z) = \left\{\beta\int_0^z t^{\beta-1}[g'(t)]^\gamma\right\}^{\frac{1}{\beta}}$$

is univalent.

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