USED OF INTERPOLATORY LINEAR POSITIVE OPERATORS FOR CALCULUS THE MOMENTS OF THE RELATED PROBABILITY DISTRIBUTIONS

by
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Abstract. In this paper using probabilistic methods for constructing linear positive operators and the Newton interpolation formula on representing a linear interpolatory positive operators by means of the factorial moments of the related probability distribution and the finite differences. By means of such representations we deduce explicit formulas for the ordinary moments of the corresponding discrete probability distributions.

Keywords: probabilistic methods, interpolatory linear positive operators, moments of the related probability distributions, factorial moments, ordinary moments.

1.Consider a sequence of two-dimensional random vectors $Y_n=\{(Y_{n1}, Y_{n2})\}$ and let $F_n(y_1, y_2; x_1, x_2)$ be the probability distribution of (Y_{n1}, Y_{n2}), where (y_1, y_2) is any point of the Euclidean space R^2 and (x_1, x_2) is a real two-dimensional parameter varying in a parameter space Ω_2, which is a subset of R^2.

We suppose that (x_1, x_2) represents the mean value of this distribution i.e.

$$x_1 = \int_{R^2} y_1 dF_n(y_1, y_2; x_1, x_2)$$

$$x_2 = \int_{R^2} y_2 dF_n(y_1, y_2; x_x, x_2)$$

If $f = f(y_1,y_2)$ is a real-valued function defined and bounded on R^2 such that the mean value of the random variable $f(Y_{n1},Y_{n2})$ exists for $n = 1,2,\ldots$, there fore this mean value can be expressed by the improper Stieltjes integral of (y_1,y_2) with respect to $F_n(y_1,y_2; x_1,x_2)$.

$$(1) \qquad E[f(Y_{n1}, Y_{n2})] = P_n(f; x_1,x_2) = \int_{R^2} f(y_1, y_2) dF_n(y_1, y_2; x_1, x_2)$$

If we suppose that the radom vector (Y_{n1},Y_{n2}) is of discrete type, one may observe that its distribution function :

$$F_n(y_1, y_2; x_1, x_2) = P[Y_{n1} \le y_1, Y_{n2} \le y_2; x_1, x_2]$$

is a step function so that $P[Y_{n1}=y_1, Y_{n2}=y_2; x_1, x_2]$ is zero at every point of R^2 except at a finite or a countable infinite such point (jump point) is taken with a positive probability (jump):

$$p_n(a_{k1}, a_{k2}) = p_n(a_{k1}, a_{k2}; x_1, x_2) = P[Y_{n1} = a_{k1}, Y_{n2} = a_{k2}; x_1,x_2]$$

satisfying the condition $\sum_k p_n(a_{k1}, a_{k2}; x_1, x_2) = 1$.

The corresponding distribution function is:

$$F_n(y_1, y_2; x_1, x_2) = \sum_{(k)} p_n(a_{k1}, a_{k2}; x_1, x_2)$$

where the summation is extended now over all points (a_{k1}, a_{k2}) such that $a_{k1} \le y_1$, $a_{k2} \le y_2$.

Consequently, in this discrete case we are able to write down the following expression for the operator (1)

$$(2) \qquad P_n(f; x_1, x_2) = \sum_{(k)} f(a_{k1}, a_{k2}) p_n(a_{k1}, a_{k2}; x_1, x_2)$$

It is easy to see that the operator $P_n(f; x_1, x_2)$ defined by (1), or in particular by (2) is a positive linear operator.

2.We shall now make use of an important method for constructing concrete operators of this useful for computing the moments of the related probability distributions.

Consider a sequence of two-dimensional random vectors $\{(X_{k1}, X_{k2}) = X_k\}$ and let us assume that the components Y_{n1}, Y_{n2} of the random vector Y_n represent the arithmetic means of the first n components X_{k1}, X_{k2} $(k = 1,2,\ldots,n)$ that is:

$$(3) \qquad Y_{nr} = \frac{1}{n}[X_{1r} + X_{2r} + \ldots + X_{nr}] \quad (r = 1,2).$$

i) Let us suppose first that the components Y_{n1} , Y_{n2} have the binomial distribution. Now referring to (2) we obtain the operator of Bernstein

$$(4) \qquad B_n(f; x_1, x_2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} x_1^{k_1} x_2^{k_2} (1 - x_1 - x_2)^{n-k_1-k_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$$

ii) If we consider that $X_{1i}, X_{2i}, \ldots, X_{ni}$ $(i=1,2)$ has a Poisson distribution with the parameters x_i $(i=1,2)$ therefore Y_{ni} have a Poisson distribution with the parameters nx_i $(i=1,2)$ and we obtain the operator

$$(5) \qquad P_n(f; x_1, x_2) = \sum_{k_1,k_2=0}^{\infty} e^{-n(x_1+x_2)} \frac{(nx_x)^{k_1} (nx_2)^{k_2}}{k_1! k_2!} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$$

which represents an extension to 2 variables of an operator studied early by Favard [3] and Szasz [15].

iii) If we presuppose that X_{1i}, X_{2i}, …, X_{ni} (i=1,2) has a geometric distribution then Y_{ni} have a Pascal distribution and we obtain the operator

(6)

$$P_n(f;x_1,x_2) = \sum_{k_1,k_2=0}^{\infty} \binom{n+k_1-1}{k_1}\binom{n+k_2-1}{k_2}(x_1x_2)^n(1-x_1)^{k_1}(1-x_2)^{k_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)$$

iv) It should be observed that if we replace x_i by $\frac{1}{1+x_i}$, (i=1,2) in formula (6), then we arrive the operator (7)

$$P_n(f;x_1,x_2) = \sum_{k_1,k_2=0}^{\infty} \binom{n+k_1-1}{k_1}\binom{n+k_2-1}{k_2}\frac{x_1^{k_1}x_2^{k_2}}{(1+x_1)^{n+k_1}(1+x_2)^{n+k_2}} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)$$

which has been considered first by Baskakov [1].

v) If the random variables X_{1i}, X_{2i}, …, X_{ni} (i=1,2) are not independent and identically distributed we obtain the operator of Stancu

$$(8) \qquad P_n^{[\alpha]}(f;x_1,x_2) = \sum_{k_1+k_2=0}^{n} W_n^{k_1,k_2}(x_x,x_2;\alpha) f\left(\frac{k_1}{n},\frac{k_2}{n}\right)$$

where

$$W_n^{k_1,k_2}(x_x,x_2;\alpha) = \frac{C_n^{k_1,k_2}}{B(n)}\prod_{\nu_1=0}^{k_1-1}(x_1+\nu_1\alpha)\prod_{\nu_2=0}^{k_2-1}(x_2+\nu_2\alpha)\prod_{\mu=0}^{n-k_1-k_2-1}(1-x_1-x_2+\mu\alpha)$$

$$B(n) = (1+\alpha)(1+2\alpha)...(1+(n-1)\alpha)\ , \quad C_n^{k_1,k_2} = \frac{n!}{k_1!k_2!(n-k_1-k_2)!}$$

3. Now we consider of an interpolation polynomial of Newton-Bierman type for two variables [8]

$$N(f;t_1,t_2) = \sum_{i_1+i_2=0}^{n} \frac{(nt_1)^{[i_1]}(nt_2)^{[i_2]}}{i_1!i_2!}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}} f(0,0)$$

where $(nt_k)^{[i_k]} = nt_k(nt_k-1)...(nt_k-i_k+1)$ (k=1,2) while

$$\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}} f(0,0) = \sum_{\nu_1=0}^{i_1}\sum_{\nu_2=0}^{i_2}(-1)^{\nu_1+\nu_2}\binom{i_1}{\nu_1}\binom{i_2}{\nu_2} f\left(\frac{i_1-\nu_1}{n},\frac{i_2-\nu_2}{n}\right)$$ represents the finite partial difference of order (i_1, i_2) of the function f, with the steps $\frac{1}{n}$ and the starting point (0,0).

With the aid of the changes of variables $nt_k=y_k$ (k=1,2) we obtain

(9) $$N\left(f;\frac{y_1}{n},\frac{y_2}{n}\right)=\sum_{i_1+i_2=0}^{n}\frac{y_1^{[i_1]}y_2^{[i_2]}}{i_1!i_2!}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}}f(0,0)$$

This polynomial satisfies the interpolating properties

$N\left(f;\frac{k_1}{n},\frac{k_2}{n}\right)=f\left(\frac{k_1}{n},\frac{k_2}{n}\right)$ for k1 = 0, 1, …, n , k2 = 0, 1, …, n-k1.

By using the formula (9) we can find for the mean value of the random variable

$N\left(f;\frac{y_1}{n},\frac{y_2}{n}\right)$ where (y_1, y_2) has the probability distribution function $F_n(y_1, y_2; x_1, x_2)$, the following representation

(10) $$\int_{R^2}N\left(f;\frac{y_1}{n},\frac{y_2}{n}\right)dF_n(y_1,y_2;x_1,x_2)=\sum_{i_1+i_2=0}^{n}\frac{m_{[i_1,i_2]}}{i_1!i_2!}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}}f(0,0)$$

in terms of the factorial moments

$$m_{[i_1,i_2]}=\int_{R^2}y_1^{[i_1]}y_2^{[i_2]}dF_n(y_1,y_2;x_1,x_2)$$

If the random vector Yn is of discrete type then

(11) $$\sum_{k_1+k_2=0}^{n}p_{n;k_1,k_2}f\left(\frac{k_1}{n},\frac{k_2}{n}\right)=\sum_{k_1+k_2=0}^{n}\frac{m_{[i_1,i_2]}}{i_1!i_2!}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}}f(0,0)$$

where $m_{[i_1,i_2]}=\sum_{k_1+k_2=0}^{n}k_1^{[i_1]}k_2^{[i_2]}p_{n;k_1,k_2}$.

4. i) With the aid of formula (11) we can give the following representation of the operators of Bernstein type (4) in terms of finite differences

(12) $$B_n(f;x_1,x_2)=\sum_{i_1+i_2=0}^{n}\frac{n^{[i_1+i_2]}}{i_1!i_2!}x_1^{i_1}x_{2n}^{i_2}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}}f(0,0)$$

which enables us to find the factorial moments

(13) $$m_{[i_1,i_2]}=n^{[i_1+i_2]}x_1^{i_1}x_2^{i_2}$$

ii) The operator $P_n(f;x_1, x_2)$ defined by (5) in terms of finite differences have the following representation

(14) $$P_n(f;z_1,z_2)=\sum_{i_1,i_2=0}\frac{z_1^{i_1}z_2^{i_2}}{i_1!i_2!}\Delta^{i_1,i_2}_{\frac{1}{n},\frac{1}{n}}f(0,0)$$

if we assume that x_i depends on n such a way for $n \to \infty$ we have $nx_i \to z_i > 0$, (i=1,2) and we obtain the factorial moments

(15) $$m_{[i_1,i_2]}=z_1^{i_1}z_2^{i_2}$$

iii) The operators of Stancu defined by (8) in terms of finite differences have the following representation (16)

$$P_n^{[\alpha]}(f;x_1,x_2)=\sum_{i_1+i_2=0}^{n}\frac{C_n^{i_1,i_2}}{B(i_1,i_2)}\prod_{\nu=1}^{2}x_\nu(x_\nu+\alpha)...(x_\nu+(i_\nu-1)\alpha)\Delta_{\frac{1}{n},\frac{1}{n}}^{i_1,i_2}f(0,0)$$

and we obtain the factorial moments

$$(17)\qquad m_{[i_1,i_2]}=\frac{n^{[i_1+i_2]}}{B(i_1+i_2)}\prod_{\nu=1}^{2}x_\nu(x_\nu+\alpha)...(x_\nu+(i_\nu-1)\alpha)$$

5. For the function $f(y_1,y_2)=n^{r_1+r_2}y_1^{r_1}y_2^{r_2}$ we have $\Delta_{\frac{1}{n},\frac{1}{n}}^{i_1,i_2}f(0,0)=\Delta^{i_1}0^{r_1}\Delta^{i_2}0^{r_2}$.

In this case the operator defined by (12) permits us to find the express of the ordinary moments

$$B_n(f;x_1,x_2)=m_{r_1,r_2}(n;x_1,x_2)$$

that is

$$(18)\qquad m_{r_1,r_2}(n;x_1,x_2)=\sum_{i_1=0}^{r_1}\sum_{i_2=0}^{r_2}\binom{n}{i_1}\binom{n-i_1}{i_2}\left(\Delta^{i_1}0^{r_1}\right)\left(\Delta^{i_2}0^{r_2}\right)x_1^{i_1}x_2^{i_2}$$

if $\Delta^m 0^r=0$ for m > r and $\binom{m}{\nu}=0$ for ν > m.

Analogous the operator defined by (14) enables us to find

$$(19)\qquad m_{r_1,r_2}(n;z_1,z_2)=\sum_{i_1=0}^{r_1}\sum_{i_2=0}^{r_2}\frac{z_1^{i_1}z_2^{i_2}}{i_1!i_2!}\left(\Delta^{i_1}0^{r_1}\right)\left(\Delta^{i_2}0^{r_2}\right)$$

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