NOMOGRAFIC REPRESENTATIONS OF SOME CANONICAL FORMS

by
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Abstract. The aim of this paper is to study some canonical forms of the equations with four and five variables. It have been found nomograms in space with coplanar points on which the equation can be nomographycally represented. The factors of anamorphosis are examined.

For the equations with three variables, the canonical forms were completely studied by R. Soreau [5] and M. d'Ocagne. In the case of the equations with four and five variables some canonical forms were study by J. Wojtowicz [5], [7], N. Kazangapov [1]. M. Mihoc [2], [4], had in view the nomographic representation of these canonical forms both by the compound plane nomograms and the nomograms in space with coplanar points.

This study is also extended to the other canonical forms for the equations with four and five variables, having in view their nomographic representation. We provide a classification of these canonical forms according to the genus of nomograms by which these equations (canonical forms) can be nomographically represented.

In the aim of the nomographic representation of these canonical forms, we intend the possibility of bringing these forms to the Soreau equations, either directly or by the anamorphosis factors.

1. THE EQUATION WITH FOUR VARIABLES

We will analyze, with respect to the nomographical order and with respect to the genus of nomogram, the canonical forms of the equations with four variables.

A. The equations of four nomographic order with real coefficients.

The general equation of this form is

$$
\begin{aligned}
&A_0 f_1(z_1)f_2(z_2)f_3(z_3)f_4(z_4)+A_1 f_2(z_2)f_3(z_3)f_4(z_4)+\\
&+A_2 f_1(z_1)f_3(z_3)f_4(z_4)+A_3 f_1(z_1)f_2(z_2)f_4(z_4)+\\
&+A_4 f_1(z_1)f_2(z_2)f_3(z_3) +B_{12} f_1(z_1)f_2(z_2)+B_{13} f_1(z_1)f_3(z_3)+\\
&+B_{14} f_1(z_1)f_4(z_4)+B_{23} f_2(z_2)f_3(z_3)+ +B_{24} f_2(z_2)f_4(z_4)+\\
&+B_{34} f_3(z_3)f_4(z_4)+C_1 f_1(z_1)+C_2 f_2(z_2)+\\
&+C_3 f_3(z_3)+C_4 f_4(z_4)+D=0.
\end{aligned}
$$

In [2], [4] we analyzed the nomograme in space by which are represented those three canonical forms founded by N. Kazangapov [1]

$$f_1(z_1)f_2(z_2)f_3(z_3)f_4(z_4)=1 \quad (1)$$

$$f_1(z_1)+f_2(z_2)+f_3(z_3)+f_4(z_4)=0 \quad (2)$$

$$f_1(z_1)f_2(z_2)f_3(z_3)+f_1(z_1)f_2(z_2)f_4(z_4)+ f_1(z_1)f_3(z_3)f_4(z_4)+$$
$$+f_2(z_2)f_3(z_3)f_4(z_4) = f_1(z_1)+f_2(z_2)+f_3(z_3)+f_4(z_4) \quad (3)$$

where z_i, i=1,4 are the variables of equations.

It is clearly that for the forms (1) and (2) are satisfied the condition of Goursat and in consequence each of them can be nomographically represented by a plane compound nomogram.

J. Wojtowicz [7] founded another equation, which we can consider it as a fourth canonical form for the equations with four variables. It is:

$$f_i(z_i) + f_j(z_j) = f_k(z_k)f_m(z_m) \quad (4)$$

where i, j, k, m =1,…, 4, $i \neq j \neq k \neq m$.

With the notations (which represent three of the disjunction equations)

$$x = f_i(z_i),\ \ y = f_j(z_j),\ \ z = f_k(z_k) \quad (5)$$

we obtain, after the substitution in (4), the fourth disjunction equation

$$x + y - zf_m(z_m) = 0. \quad (6)$$

From (5) and (6) we obtain the Soreau equation in space

$$\begin{vmatrix} 1 & 0 & 0 & -f_i \\ 0 & 1 & 0 & -f_j \\ 0 & 0 & 1 & -f_k \\ 1 & 1 & -f_m & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & f_i & 1 \\ 0 & 1 & f_j & 1 \\ 0 & 0 & f_k & 1 \\ \frac{1}{2-f_m} & \frac{f_m}{f_m-2} & 0 & 1 \end{vmatrix} = 0 \quad (7)$$

Here we perform the elementary transformation to the Massau determinant from (7).

We fail to write the argument of functions which are in equation (7) having in view the simpleness of the formula's writing. In such way we proceed in those which follows.

From (7) we infer the equations of the elements of nomogram in space with coplanar points by which the equation (4) is represented. They are: three straight line scales which correspond to the variables z_i, z_j, z_k; the first one is situated on the axis OZ and the last two are situated on two parallel straight line with OZ. The scale (z_j) is in plane XOZ and the scale (z_k) in plane YOZ. The fourth scale (z_m) is situated on a plane curve in XOY.

Therefore, the canonical form (4) is nomographically represented by a nomogram in space with coplanary points of genus one.

ii) We can increase the genus of nomogram that corresponds to the equation (4) if we multiply this equation with an anamorphosis factor, e.g. (f_i-f_j)

$$(f_i - f_j)[f_i + f_j - f_k f_m]=0 \quad (8)$$

or, after that we order it by the powers of function f_i

$$f_i^2 - f_i f_k f_m + f_j(f_k f_m - f_j)=0 \tag{9}$$

With the notations

$$x = f_k f_m,\ \ y = f_j f_k f_m - f_j^2,\ \ z = f_k + f_m$$

we obtain the disjunction equations

$$\begin{cases} f_i^2 - f_i x + y = 0 \\ f_j x - f_j^2 - y = 0 \end{cases} \qquad \begin{cases} f_k^2 - f_k z + x = 0 \\ f_m^2 - f_m z + x = 0 \end{cases} \tag{10}$$

and, after the elementary transformations, the Soreau equation

$$\begin{vmatrix} -f_i & 1 & 0 & f_i^2 \\ -f_j & 1 & 0 & f_j^2 \\ 1 & 0 & -f_k & f_k^2 \\ 1 & 0 & -f_m & f_m^2 \end{vmatrix} = \begin{vmatrix} 1/f_i & 1 & -f_i & 1 \\ 1/f_j & 1 & -f_j & 1 \\ 0 & f_k & f_k^2 & 1 \\ 0 & f_m & f_m^2 & 1 \end{vmatrix} = 0 \tag{11}$$

The scales of the variables z_i, z_j have a common support, a hyperbole, XZ = -1, situated in plane XOZ. The scales of the variables z_k, z_m have also the same support, a parabola situated in plane YOZ, Z = Y².

Therefore, the equation (4) can be also nomographically represented by a nomogram in space of genus four.

iii) It is clearly that the equation (4) satisfies the Goursat condition of separation the variables and then it can be nomographically represented by a plane compound nomogram from two nomogram with alignment points with a "dumb" common scale. One of these nomogram has all three scales parallel and the other has the scales in "N" position.

Remark. In the case of building the nomogram for a concrete equation the modulus of the scales and the dimension of nomogram must necessarily appear.

B. The equations of five nomographical order

i) They have the general forms

$$\begin{aligned} &[A_0 f_1 f_2 f_3 + A_1(f_1 f_2 + f_1 f_3 + f_2 f_3) + A_2(f_1 + f_2 + f_3) + A_3] f_4 \ + \\ &+[B_0 f_1 f_2 f_3 + B_1(f_1 f_2 + f_1 f_3 + f_2 f_3) + B_2(f_1 + f_2 + f_3) + B_3] g_4 + \\ &+ [C_0 f_1 f_2 f_3 + C_1(f_1 f_2 + f_1 f_3 + f_2 f_3) + C_2(f_1 + f_2 + f_3) + C_3] h_4 = 0. \end{aligned} \tag{12}$$

In order to obtain the Soreau equation we consider the notations

$$x = f_1 + f_2 + f_3,\ y = f_1 f_2 + f_1 f_3 + f_2 f_3,\ z = f_1 f_2 f_3 \tag{13}$$

and by replacing in (12) we will have the linear equation in respect to x, y, z

$$\begin{aligned} &(A_0 z + A_1 y + A_2 x + A_3) f_4 + (B_0 z + B_1 y + B_2 x + B_3) g_4 + \\ &+ (C_0 z + C_1 y + C_2 z + C_3) = 0 \end{aligned}$$

or

$$(A_2 f_4 + B_2 g_4 + C_2)x + (A_1 f_4 + B_1 g_4 + C_1)y +$$

$$+ (A_0 f_4 + B_0 g_4 + C_0)z + (A_3 f_4 + B_3 g_4 + C_3) = 0. \tag{14}$$

The coefficients of (14) depend only of the variable z_4.

From (13) we conclude

$$f_i^3 - f_i^2 x + f_i y - z = 0, \quad i = 1,3 \tag{15}$$

and therefore we obtain after the ordinary operations, the Soreau equation

$$\begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 \\ f_2 & f_2^2 & f_2^3 & 1 \\ f_3 & f_3^2 & f_3^3 & 1 \\ -\dfrac{A_1 f_4 + B_1 g_4 + C_1}{A_0 f_4 + B_0 g_4 + C_0} & \dfrac{A_2 f_4 + B_2 g_4 + C_2}{A_0 f_4 + B_0 g_4 + C_0} & -\dfrac{A_3 f_4 + B_3 g_4 + C_3}{A_0 f_4 + B_0 g_4 + C_0} & 1 \end{vmatrix} = 0 \tag{16}$$

The parametrical equations of the scales (z_i), i=1,3 are

$X = f_i(z_i)$, $Y = f_i^2(z_i)$, $Z = f_i^3(z_i)$, $i=1,3$

and the analytical equations of the support curve (curve distorted in space) are

$Y = X^2$ and $Z = X^3$

The parametrical equations of the support scales (z_4) are the rational functions of the variables z_4

$$X = -\frac{A_1 f_4 + B_1 g_4 + C_1}{A_0 f_4 + B_0 g_4 + C_0}, \quad Y = \frac{A_2 f_4 + B_2 g_4 + C_2}{A_0 f_4 + B_0 g_4 + C_0}, \quad Z = -\frac{A_3 f_4 + B_3 g_4 + C_3}{A_0 f_4 + B_0 g_4 + C_0}$$

We proved that the equation with four variables of the five nomographic order five, (12), can be represented by a nomogram in space with coplanary points, having three scales (z_i), i=1,3, situated on the same curviligne support (a distorted curve in space) $Y=X^2$, $Z=X^3$ and the fourth scale situated on another distorted curve in space.

ii) We have here two canonical forms which we will denote, analogous to those with three variables, the Cauchy form and the Clark form in space

$$f_1 f_4 + f_2 g_4 + f_3 h_4 = 0 \tag{17}$$

$$f_1 f_2 f_3 f_4 + (f_1 f_2 + f_1 f_3 + f_2 f_3) g_4 + (f_1 + f_2 + f_3) h_4 = 0 \tag{18}$$

For the equation (17) we have the equation of disjunction

$x=f_1$, $y=f_2$, $z=f_3$, $f_4 x + g_4 y + f_3 z = 0$

and Soreau equation

$$\begin{vmatrix} 1 & 0 & 0 & f_1 \\ 0 & 1 & 0 & f_2 \\ 0 & 0 & 1 & f_3 \\ f_4 & g_4 & h_4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & f_1 & 1 \\ 1 & 0 & f_2 & 1 \\ 0 & 1 & f_3 & 1 \\ \dfrac{g_4}{f_4 + g_4 + h_4} & \dfrac{h_4}{f_4 + g_4 + h_4} & 0 & 1 \end{vmatrix} = 0 \tag{19}$$

Therefore, we conclude that the equation (17) is represented by a nomogram in space with coplanary points of genus one with three parallel straight line scales (z_{i}), i=1,3. The fourth scale is situated on the plane curve in plane XOY.

For the Clark canonical form in space with the notations (13) we find the disjunction equations

$$f_i^3 - f_i^2 x + f_i y - z = 0, \quad f_4 z + g_4 y + h_4 x = 0 \tag{20}$$

which conduct to the Soreau equation

$$\begin{vmatrix} -f_1^2 & f_1 & -1 & f_1^3 \\ -f_2^2 & f_2 & -1 & f_2^3 \\ -f_3^2 & f_3 & -1 & f_3^3 \\ h_4 & g_4 & f_4 & 0 \end{vmatrix} = \begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 \\ f_2 & f_2^2 & f_2^3 & 1 \\ f_3 & f_3^2 & f_3^3 & 1 \\ -\frac{g_4}{f_4} & -\frac{h_4}{f_4} & 0 & 1 \end{vmatrix} = 0 \tag{21}$$

The nomogram in space with coplanary points for the equation (18) is of genus four. All its three scales (z_i), i=1,3 are situated on a distorted curve in space: $Y=X^2$, $Z=X^3$ and the fourth scale (z_4) is situated on a plane curve in XOY.

2. THE EQUATIONS WITH FIVE VARIABLES

We study the canonical forms of six nomographic order

$$f_1(z_1)f_{45}(z_4,z_5)+f_2(z_2)g_{45}(z_4,z_5)+f_3(z_4)h_{45}(z_4,z_5)=0 \tag{22}$$

$$\begin{aligned} & f_1(z_1)f_2(z_2)f_3(z_3)f_{45}(z_4,z_5)+ \\ & +[f_1(z_1)f_2(z_2)+f_1(z_1)f_3(z_3)+f_2(z_2)f_3(z_3)]g_{45}(z_4,z_5)+ \\ & +[f_1(z_1)+f_2(z_2)+f_3(z_3)]h_{45}(z_4,z_5)=0 \end{aligned} \tag{23}$$

and with the notations

$$x=f_1(z_1), \quad y=f_2(z_2), \quad z=f_3(z_3)$$

we obtain, by replacing in (22)

$$f_{45}(z_4,z_5)x+g_{45}(z_4,z_5)y+h_{45}(z_4,z_5)z = 0$$

and also the Soreau equation

$$\begin{vmatrix} 1 & 0 & 0 & f_1 \\ 0 & 1 & 0 & f_2 \\ 0 & 0 & 1 & f_3 \\ f_{45} & g_{45} & h_{45} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & f_1 & 1 \\ 1 & 0 & f_2 & 1 \\ 0 & 1 & f_3 & 1 \\ \frac{g_{45}}{f_{45}+g_{45}+h_{45}} & \frac{h_{45}}{f_{45}+g_{45}+h_{45}} & 0 & 1 \end{vmatrix} = 0 \tag{24}$$

In particular, if the variable z_5 are real constant, we obtain from (22) the form (17) (after we modify properly the functions of the variable z_4).

The corresponding nomogram to the equation (22) is one in space with coplanary points. Its elements are three scales (z_i), i=1,3 situated on three parallel straight lines with OZ axis and a binary net (z_4,z_5) consisting of two families of marked curves in the XOY plane.

With the notations

$$x=f_1(z_1)+f_2(z_2)+f_3(z_3),$$

$$y=f_1(z_1)f_2(z_2)+f_1(z_1)f_3(z_3)+f_2(z_2)f_3(z_3),$$
$$z=f_1(z_1)f_2(z_2)f_3(z_3)$$

we will find the disjonction equations for the equation (23), e.g.

$$f_i^3(z_i)-f_i^3(z_i)x+f_i(z_i)y-z=0,$$
$$f_{45}(z_4,z_5)z+g_{45}(z_4,z_5)y+h_{45}(z_4,z_5)x=0 \qquad (25)$$

Then the Soreau equation is immediate

$$\begin{vmatrix} f_1(z_1) & f_1^2(z_1) & f_1^3(z_1) & 1 \\ f_2(z_2) & f_2^2(z_2) & f_2^3(z_2) & 1 \\ f_3(z_3) & f_3^2(z_3) & f_3^3(z_3) & 1 \\ -\dfrac{g_{45}(z_4,z_5)}{f_{45}(z_4,z_5)} & -\dfrac{h_{45}(z_4,z_5)}{f_{45}(z_4,z_5)} & 0 & 1 \end{vmatrix} = 0 \qquad (26)$$

Therefore the nomogram which correspond to the canonical form (23) is a space nomogram with coplanary points. It contains three marked scales (z_{i}), i=1,3 situated on the same support curviligne in space and a binary net (formed by two marked curves, one of the family z_4 and the other of family z_5). This binary net is situated in the XOY plane.

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