ON THE TWO DIMENSIONAL SPLINE INTERPOLATION OF HERMITE-TYPE

by
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Abstract. In [5] the authors gave the definition of the interpolating cubic spline of Hermite-type in two variables: this is a polynomial of degree three in both variables and interpolates the function values and the values of the following derivates $D^{0,1}, D^{1,0}, D^{1,1}$ at the knots of a given rectangular subdivision. We give two similar constructions but we use only the function values and the values of the derivates $D^{0,1}, D^{1,0}$ of the unknown function at the knots. WE can prove similar estimates, but in order to compute the values of our spline function we need fewer algebric operations .

In what follows we fix the region $\Omega = [a,b] \times [c,d]$ and a subdivision $a = x_0 < x_1 < ... < x_N = b, c = y_0 < y_1 < ... < y_M = d$ of Ω. Let $\Omega_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and $h_i = x_{i+1} - x_i, l_j = y_{j+1} - y_j$. If u is a function defined on Ω, then $u_{i,j}^{(r,s)} = D^{r,s} u(x_i, y_j)$, where the differential operator $D^{r,s}$ has the obvious meaning.

2. We are to determine the spline function S of Hermite-type of two variables with the following properties:

(i) On the rectangle $\Omega_{i,j}$ we have

$$S(x,y) = S_{i,j}(x,y) = \sum_{\substack{\alpha,\beta=0 \\ \alpha+\beta\le 4}}^{3} A_{i,j}^{(\alpha,\beta)} (x - x_i)^{\alpha} (y - y_j)^{\beta}$$

(ii) At the knots we have

$$D^{r,s} S(x_i, y_i) = u_{i,j}^{(r,s)}, r,s = 0,1; r+s \le 1 \ , \ i = 0,1,...,N; j = 0,1,...,M$$

The following theorem can be verified by an easy calculation.

Theorem 2.1. If $A_{i,j}^{(2,2)} = 0,$ then there exists a unique spline function satisfying conditions (i), (ii). The function S is continuous on Ω.

By the substitution $t = (x - x_i)/h_i, v = (y - y_j)/l_j$ we can express the spline function S in the following form for (x,y)$\in \Omega_{i,j}$

$$S_{i,j}(x,y) = (1-v)\left\{\Phi_1(t)u_{i,j} + \Phi_2(t)u_{i+1,j} + \Phi_3(t)h_i u_{i,j}^{(1,0)} + \Phi_4(t)h_i u_{i+1,j}^{(1,0)}\right\} +$$
$$+ v\left\{\Phi_1(t)u_{i,j+1} + \Phi_2(t)u_{i+1,j+1} + \Phi_3(t)h_i u_{i,j+1}^{(1,0)} + \Phi_4(t)h_i u_{i+1,j+1}^{(1,0)}\right\} +$$
$$+ (1-v)v\{v[(1-t)(u_{i,j+1} - u_{i,j} - l_j u_{i,j+1}^{(0,1)} + t(u_{i+1,j+1} - u_{i+1,j} - l_j u_{i+1,j+1}^{(0,1)})] + \tag{2.1.}$$
$$+ (1-v)[(1-t)(l_j u_{i,j}^{(0,1)} - (u_{i,j+1} - u_{i,j})) + t(l_j u_{i+1,j}^{(0,1)} - (u_{i+1,j+1} - u_{i+1,j}))]\}$$

where

$$\Phi_1(t) = (1-t)^2(1+2t), \Phi_2(t) = t^2(3-2t), \Phi_3(t) = t(1-t)^2, \Phi_4(t) = -t^2(1-t).$$

Using this form, in order to compute S(x,y) at the point (x,y) in $\Omega_{i,j}$ we need 56 algebric operations: 34 additions and 22 multiplications.

Theorem 2.2. If $u \in C^{1,1}(\Omega)$, then

$$\|u(x,y) - S(x,y)\| \le \frac{3}{8}\bar{h}\omega(D^{1,0}u) + \frac{1}{2}\bar{l}\omega(D^{0,1}u)$$

Further, if $u \in C^{2,2}(\Omega)$, then

$$\|u(x,y) - S(x,y)\| \le \frac{1}{32}\bar{h}^2\omega(D^{2,0}u) + \frac{1}{3\sqrt{3}}\bar{l}^2\omega(D^{0,2}u)$$

Proof. First we suppose that $u \in C^{1,1}(\Omega)$ and let $(x,y) \in \Omega_{i,j}$. Using the theorem for the interpolating cubic splines of Hermite-type in one variable, we have:

$$\left\|\Phi_1(t)u_{i,p} + \Phi_2(t)u_{i+1,p} + \Phi_3(t)h_i u_{i,p}^{(1,0)} + \Phi_4(t)h_i u_{i+1,p}^{(1,0)} - u(x,y_p)\right\| \le \frac{3}{8}\bar{h}\omega(D^{1,0}u) \quad (p = j, j+1)$$

By fixing the first variable and applying the Lagrange theorem in the second variable, for the third term of (2.1.) we obtain:

$$(1-v)vl_j\{v[(1-t)(D^{0,1}u(x_i,\bar{\eta}) - u_{i,j+1}^{(0,1)}) + t(D^{0,1}u(x_{i+1},\bar{\bar{\eta}}) - u_{i+1,j+1}^{(0,1)})] +$$
$$+ (1-v)[(1-t)(u_{i,j}^{(0,1)} - D^{0,1}u(x_i,\bar{\eta})) + t(u_{i+1,j}^{(0,1)} - D^{0,1}u(x_{i+1},\bar{\bar{\eta}}))]\}$$

where $\bar{\eta}, \bar{\bar{\eta}} \in (y_j y_{j+1})$. Hence, using this and the form (2.1.), we have:

$$\|u(x,y) - S(x,y)\| \le \frac{3}{8}\bar{h}\omega(D^{1,0}u) + \|(1-v)[u(x,y_j) - u(x,y)] + v[u(x,y_{j+1}) - u(x,y)]\| +$$
$$+ (1-v)vl_j\omega(D^{0,1}u) \le \frac{3}{8}\bar{h}\omega(D^{1,0}u) + \frac{1}{2}\bar{l}\omega(D^{0,1}u)$$

In the case $u \in C^{2,2}(\Omega)$ we fix one variable of the function u and apply the Taylor formula of the second order and the estimation

$$\left\|\Phi_1(t)u_{i,p} + \Phi_2(t)u_{i+1,p} + \Phi_3(t)h_i u_{i,p}^{(1,0)} + \Phi_4(t)h_i u_{i+1,p}^{(1,0)} - u(x,y_p)\right\| \le \frac{1}{32}\bar{h}^2\omega(D^{2,0}u)$$

By the Taylor formula and the continuity of $D^{0,2}$, we can express the third term of (2.1.) in the form:

$$-\frac{1}{2}(1-v)vl_j^2\{v[(1-t)D^{0,2}u(x_i,\tilde{\eta})+tD^{0,2}u(x_{i+1},\tilde{\tilde{\eta}})]+(1-v)[(1-t)D^{0,2}u(x_i,\bar{\eta})+tD^{0,2}u(x_{i+1},\bar{\bar{\eta}})]\}=$$

$$=-\frac{1}{2}(1-v)vl_j^2\{vD^{0,2}u(\bar{\xi},\bar{\zeta})+(1-v)D^{0,2}u(\tilde{\xi},\tilde{\zeta})\}=-\frac{1}{2}(1-v)vl_j^2D^{0,2}u(\xi,\zeta)$$

where $(\xi,\zeta)\in\Omega_{i,j}$.

By these considerations we have:

$$\left\|u(x,y)-S_{i,j}(x,y)\right\|\leq\frac{1}{32}\bar{h}^2\,\omega\left(D^{2,0}u\right)+$$

$$+\left\|(1-v)\left[u(x,y_j)-u(x,y)\right]+v\left[u(x,y_{j+1})-u(x,y)\right]-\frac{1}{2}(1-v)vl_j^2D^{0,2}u(\xi,\zeta)\right\|\leq$$

$$\leq\frac{1}{32}\bar{h}^2\,\omega\left(D^{2,0}u\right)+\frac{1}{2}(1-v)v(1+v)\bar{l}^2\,\omega\left(D^{0,2}u\right)\leq\frac{1}{32}\bar{h}^2\,\omega\left(D^{2,0}u\right)+\frac{\sqrt{3}}{9}\bar{l}^2\,\omega\left(D^{0,2}u\right),$$

which proves our statement .

3. We are to determine the spline function S of Hermite-type of two variables with the following properties:

(i') On the rectangle $\Omega_{i,j}$ we have:

$$S(x,y)=S_{i,j}(x,y)=\sum_{\substack{\alpha,\beta=0\\ \alpha+\beta\leq 3}}^{2}A_{i,j}^{(\alpha,\beta)}(x-x_i)^{\alpha}(y-y_j)^{\beta}$$

(ii') At the knots we have $(i=0,...,N;\,j=0,...,M)$

$$S_{i,j}(x_p,y_q)=u_{p,q}$$

$$D^{1,0}S_{i,j}(x_i,y_q)=u_{i,q}^{(1,0)}$$

$$D^{0,1}S_{i,j}(x_p,y_j)=u_{p,j}^{(0,1)}$$

where p = i, i+1 ; q = j, j+1.

The following theorem can be verified by an easy calculation .

Theorem 3.1. There exists a unique spline function S with the properties (i'), (ii') and it is continuous on Ω.

By the substitution $t=\left(x-x_i\right)/h_i,\ v=\left(y-y_j\right)/l_j$ we can express the spline function S in the following form for $(x,y)\in\Omega_{i,j}$:

$$S_{i,j}(x,y)=(1-v)\{u_{i,j}+t[h_i u_{i,j}^{(1,0)}+t(u_{i+1,j}-u_{i,j}-h_i u_{i,j}^{(1,0)})]\}+v\{u_{i,j+1}+t[h_i u_{i,j+1}^{(1,0)}+$$
$$+t(u_{i+1,j+1}-u_{i,j+1}-h_i u_{i,j+1}^{(1,0)})]\}+(1-v)v\{(l_j u_{i,j}^{(0,1)}-u_{i,j+1}+u_{i,j})(1-t)+(l_j u_{i+1,j}^{(0,1)}-u_{i+1,j+1}-u_{i+1,j})t\}. \quad (3.1.)$$

In order to compute the value S(x,y) = $S_{i,j}$ (x,y) at the print $(x,y)\in\Omega_{i,j}$ we need 16 multiplications and 21 additions.

Lemma 3.1. Let f be differentiable on [a,b], and let for $x\in[x_i,x_{i+1}]$.

$$S_i(x)=f(x_i)+f'(x_i)(x-x_i)+\frac{1}{h_i^2}[f(x_{i+1})-f(x_i)-f'(x_i)h_i](x-x_i)^2.$$

If $f\in C^1[a,b]$, *then* $|f(x)-S_i(x)|<\frac{2}{3\sqrt{3}}\bar{h}\omega(f')$

If $f\in C^2[a,b]$, *then* $|f(x)-S_i(x)|\le\frac{46}{1000}\bar{h}^2\omega(f'')$.

Proof. By the substitution $t={(x-x_i)}/{h_i}$, $f_i=f(x_i)$, $f_i'=f'(x_i)$ we have

$$R(x)=S_i(x)-f(x)=f_i(1-t^2)+f_{i+1}t^2+h_i f_i'(t-t^2)-f(x).$$

For $f\in C^1[a,b]$ the Lagrange theorem for the interval $[x_i,x_{i+1}]$ gives:

$$R(x)=-f'(\xi_1)h_i t(1-t^2)+f'(\xi_2)h_i(1-t)t^2+h_i f_i' t(1-t)=h_i t(1-t)[-f'(\xi_1)(1+t)+$$
$$+(f'(\xi_2)t+f_i')]=h_i t(1-t)(1+t)[f'(\xi)-f'(\xi_1)],$$

where $\xi_1,\xi_2,\xi\in(x_i,x_{i+1})$. Hence we have

$$|S_i(x)-f(x)|\le h_i t(1-t)(1+t)\omega_i(f')\le h_i\frac{2}{3\sqrt{3}}\omega_i(f')\le\bar{h}\frac{2}{3\sqrt{3}}\omega(f')$$

For $f\in C^2[a,b]$ we substitute the values f_i, f_{i+1}, f_i' from the Taylor formula at the point $x=x_i+th_i$ and get

$$R(x)=\int_{x_i}^{x}(1-t)[(1+t)(v-x_i)-th_i]f''(v)dv+\int_x^{x_{i+1}}t^2(x_{i+1}-v)f''(v)dv$$

In both integrals we let $\tau=(v-x_i)/h_i$. Then

$$R(x)=h_i^2\left[\int_0^t\Psi_1(t_1\tau)f''(x_i+\tau h_i)d\tau+\int_t^1\Psi_2(t,\tau)f''(x_i+\tau h_i)d\tau\right],$$

where $\Psi_1(t,\tau)=(1-t)[(1+t)\tau-t]$, $\Psi_2(t,\tau)=t^2(1-\tau)$.

The function $\Psi_1(t,\tau)$ changes sign only at the point $\tau=\tau^*=t/(1+t)\in[0,t]$, hence by applying the Mean Value Theorem on the intervals $[0,\tau^*],[\tau^*,t]$ we have:

$$\int_0^t \Psi_1(t,\tau) f''(x_i+\tau h_i)d\tau = f''(\xi)\int_0^{\tau^*}\Psi_1(t,\tau)d\tau + f''(\eta)\int_{\tau^*}^{t}\Psi_1(t,\tau)d\tau = -f''(\xi)\frac{t^2(1-t)}{2(1-t)} + f''(\eta)\frac{t^4(1-t)}{2(1+t)} =$$

$$= f''(\xi)\frac{t^2}{2} + f''(\eta)\frac{t^4(1-t)}{2(1+t)},$$

where $\xi,\eta \in [x_i, x_{i+1}]$. The function $\Psi_2(t,\tau)$ does not change sign for $\tau\in[t,1]$; hence $\int_t^1 \Psi_2(t,\tau) f''(x_i+\tau h_i)d\tau = f''(\zeta)\frac{t^2(1-t)^2}{2}$ where $\zeta\in[x_i,x_{i+1}]$. By these considerations we have:

$$R(x) = h_i^2\frac{1-t}{2}\left\{\left(f''(\varsigma)t^2(1-t) + f''(\xi)\frac{t^4}{1+t}\right) - f''(\eta)\frac{t^2}{1+t}\right\} = h_i^2\frac{(1-t)t^2}{2(1+t)}\left[f''(\bar\eta) - f''(\eta)\right],$$

where $\bar\eta\in[x_i,x_{i+1}]$, and we have used that $\operatorname{sgn} t^2(1-t) = \operatorname{sgn}\frac{t^4}{1+t}$, and f'' is continuous.Finally,

$|R(x)| \le h_i^2\frac{5\sqrt5-11}{4}\left|f''(\bar\eta) - f''(\eta)\right| \le 0{,}046 h_i^2\omega_i(f'') \le \frac{46}{1000}\bar h\omega(f'')$ and lemma is proved.

Theorem 3.2. If $u\in C^{1,1}(\Omega)$, then

$$\|u(x,y)-S(x,y)\| \le \frac{2}{3\sqrt3}\bar h\omega(D^{1,0}u) + \frac12\bar l\omega(D^{0,2}u)$$

If $u\in C^{2,2}(\Omega)$, then

$$\|u(x,y)-S(x,y)\| \le 0{,}046\bar h^2\omega(D^{2,0}u) + \frac14\bar l^2\omega(D^{0,2}u)$$

Proof. We suppose that $u\in C^{1,1}(\Omega)$. If we use the form (3.1.) of $S_{i,j}(x,y)$ and apply the lemma 3.1. for $u(x,y_j)$ and $u(x,y_{j+1})$ then we obtain for $(x,y)\in\Omega_{i,j}$.

$$|S_{i,j}(x,y)-u(x,y)| \le (1-v+v)\frac{2}{3\sqrt3}\bar h\omega(D^{1,0}u) + |(1-v)[u(x,y_j)-u(x,y)] + v[u(x,y_{j+1})-u(x,y)]| +$$

$$+(1-v)v\left|(u_{i,j+1}-u_{i,j}-l_j u_{i,j}^{(0,1)})(1-t) + (u_{i+1,j+1}-u_{i+1,j}-l_j u_{i+1}^{(0,1)})t\right|.$$

Applying the Lagrange theorem for the second and third term we get.

$$\left|S_{i,j}(x,y)-u(x,y)\right| \le \frac{2}{3\sqrt{3}}\bar{h}\omega(D^{1,0}u)+\left|-l_j(1-v)vD^{0,1}u(x,\bar{\eta})+l_jv(1-v)D^{0,1}u(x,\bar{\bar{\eta}})\right|+$$

$$+(1-v)vl_j\left|(D^{0,1}u(x_i,\bar{\zeta})-u_{i,j}^{(0,1)})(1-t)+(D^{0,1}u(x_{i+1},\bar{\bar{\zeta}})-u_{i+1,j}^{(0,1)}t\right| \le \frac{2}{3\sqrt{3}}\bar{h}\omega(D^{1,0}u)+$$

$$+2v(1-v)\bar{l}\omega(D^{0,1}u) \le \frac{2}{3\sqrt{3}}\bar{h}\omega(D^{1,0}u)+\frac{1}{2}\bar{l}\omega(D^{0,1}u),$$

$\bar{\eta},\bar{\bar{\eta}},\bar{\zeta},\bar{\bar{\zeta}} \in [y_j, y_{j+1}]$.

If $u \in C^{2,2}(\Omega)$ then the above considerations give:

$$\left|S_{i,j}(x,y)-u(x,y)\right| \le 0{,}046\bar{h}^2\omega(D^{2,0}u)+(1-v)v\left|-l_jD^{0,1}u(x,y_j)-\frac{1}{2}l_j^2vD^{0,2}u(x,\eta_j)+l_jD^{0,1}u(x,u_{j+1})-\right.$$

$$\left.-\frac{1}{2}l_j^2(1-v)D^{0,2}u(x,\eta_{j+1})-\frac{1}{2}l_j^2(1-t)D^{0,2}u(x_i,\tilde{\eta})-\frac{1}{2}l_j^2tD^{0,2}u(x_i\tilde{\tilde{\eta}})\right|=0{,}046\bar{h}^2\omega(D^{2,0}u)+$$

$$+(1-v)vl_j^2\left|D^{0,2}u(x,\eta)-D^{0,2}u(x_i,\bar{\eta})\right| \le 0{,}046\bar{h}^2\omega(D^{0,2}u)+\frac{1}{4}\bar{l}^2\omega(D^{0,2}u),$$

where $\eta_j, \eta_{j+1}, \tilde{\eta}, \tilde{\tilde{\eta}}, \eta, \bar{\eta} \in [y_j, y_{j+1}]$, and our theorem is proved.

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