

## ON THE SET-THEORETICAL YANG-BAXTER EQUATION

by  
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**Introduction.** The Yang-Baxter equation first appeared in theoretical physics. Afterwards, it proved to be important also in knot theory, quantum groups, etc. Many authors gave classes of solutions for the Yang-Baxter equation, but finding all solution in an arbitrary dimension ( $n > 2$ ) is an unsolved problem. For more details see [3] and [4]. For author's contribution in the field see, for example, [6].

We present the set-theoretical Yang-Baxter equation in this paper.

In section 1, we give with the terminology. In section 2, we present a new construction of solutions from boolean algebras. In section 3, we present solutions for the set-theoretical Yang-Baxter equation from group actions on a set. The last section is about commentaries and conclusions.

### Preliminaries

Let  $S$  be a set. Let  $R : SxS \rightarrow SxS$ ,  $R(u, v) = (u', v')$  be a function.

In this paper we use the following notations:

$$R_{12} : SxSxS \rightarrow SxSxS, \quad R_{12}(u, v, w) = (u', v', w)$$

$$R_{23} : SxSxS \rightarrow SxSxS, \quad R_{23}(u, v, w) = (u, v', w')$$

$$R_{13} : SxSxS \rightarrow SxSxS, \quad R_{13}(u, v, w) = (u', v, w')$$

We will study the following equation:

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23} \quad (1)$$

**Problem 1.** (*hard*) Find functions  $R : SxS \rightarrow SxS$ , such that

- i)  $R$  bijection (invertible).
- ii)  $R$  verifies the equation (1)

**Example.**  $T : SxS \rightarrow SxS$ ,  $T(x, y) = (y, x)$ , is a solution for problem 1.

(We leave this as an exercise for high school students.)

**Problem 2.**(*very hard*) Find all functions  $R : S \times S \rightarrow S \times S$ , such that the conditions i) and ii) from above (problem 1) are verified.

**Remark 1.1** The equation (1) is called the Yang-Baxter equation by some authors (see [3]); some other authors call it the braid equation (see [4]).

The following equation is called the quantum Yang-Baxter equation (*QYBE*):

$$R_{12} \circ R_{13} R_{23} = R_{23} \circ R_{13} \circ R_{12} \quad (2)$$

**Remark 1.2** The following equivalence holds:

$R$  is a solution of (1)  $\Leftrightarrow R \circ T$  is a solution of (2)

(*We leave the proof of this equivalence as an exercise for high school students.*)

### Solutions for the set-theoretical Yang-Baxter equation from boolean algebras

In this section we present a new method to construct solutions for the set-theoretical Yang-Baxter equation from boolean algebras.

**Theorem 2.1.** Let  $(A, \vee, \wedge, 0, 1, ')$  be a boolean algebra. Then

$R(A, B) = (A \vee B, A \wedge B)$  is a solution of the braid equation.

**Proof:**

$$R_{12}(A, B, C) = (A \vee B, A \wedge B, C)$$

$$R_{23} \circ R_{12}(A, B, C) = (A \vee B, (A \wedge B) \vee C, (A \wedge B) \wedge C)$$

$$R_{12} \circ R_{23} \circ R_{12}(A, B, C) =$$

$$= ((A \vee B) \vee (A \wedge B) \vee C, (A \vee B) \wedge ((A \wedge B) \vee C), A \wedge B \wedge C) =$$

$$= (A \vee B \vee C, (A \wedge B) \vee (A \wedge C) \vee (B \wedge C), A \wedge B \wedge C)$$

$$R_{23}(A, B, C) = (A, B \vee C, B \wedge C)$$

$$R_{12} \circ R_{23}(A, B, C) = (A \vee B \vee C, A \wedge (B \vee C), B \wedge C) =$$

$$= (A \vee B \vee C, (A \wedge B) \vee (A \wedge C), B \wedge C)$$

$$R_{23} \circ R_{12} \circ R_{23}(A, B, C) = (A \vee B \vee C, (A \wedge B) \vee (A \wedge C) \vee (B \wedge C), A \wedge B \wedge C)$$

**Remark 2.2.** Let  $(A, \vee, \wedge, 0, 1, ')$  be a boolean algebra. Then  $R(A, B) = (A', B')$  is an invertible solution of the braid equation. (*We leave the proof for the reader.*)

### Solutions for the set-theoretical Yang-Baxter equation from group actions on a set

**Theorem 3.1** (from [5]) Let  $G$  be a group,

$\xi : GxG \rightarrow G, (u, v) \mapsto \xi^{(u)}v$  left action  
 $\eta : GxG \rightarrow G, (u, v) \mapsto u\eta^{(v)}$  right action

satisfying the following compatibility condition:

$$uv = \left( \xi^{(u)}v \right) \left( u\eta^{(v)} \right) \quad (3)$$

Then  $\sigma : GxG \rightarrow GxG, \sigma(u, v) = \left( \xi^{(u)}v, u\eta^{(v)} \right)$  is a solution for (1).

Proof:

$$\begin{aligned} (u, v, w) &\xrightarrow{\sigma_{12}} \left( \xi^{(u)}v, u\eta^{(v)}, w \right) \xrightarrow{\sigma_{23}} \left( \xi^{(u)}v, \xi^{(u\eta^{(v)})}w, u\eta^{(v)}\eta^{(w)} \right) \xrightarrow{\sigma_{12}} \\ &\xrightarrow{\sigma_{12}} \left( \xi^{(u)}v, \xi^{(u\eta^{(v)})}w, \left( \xi^{(u)}v \right) \eta^{(u\eta^{(v)})} \right) \xrightarrow{\sigma_{12}} \left( \xi^{(u)}v, \xi^{(u\eta^{(v)})}w, u\eta^{(v)}\eta^{(w)} \right) \stackrel{not.}{=} (u_1, v_1, w_1) \\ (u, v, w) &\xrightarrow{\sigma_{23}} \left( u, \xi^{(v)}w, v\eta^{(w)} \right) \xrightarrow{\sigma_{12}} \left( \xi^{(u)}\xi^{(v)}w, u\eta^{(v)}\eta^{(w)}, v\eta^{(w)} \right) \xrightarrow{\sigma_{23}} \\ &\xrightarrow{\sigma_{23}} \left( \xi^{(u)}\xi^{(v)}w, \xi^{(u\eta^{(v)})}w, \left[ v\eta^{(w)} \right] u\eta^{(v)}\eta^{(w)} \right) \stackrel{not.}{=} (u_2, v_2, w_2) \end{aligned}$$

We first check that  $u_1 = u_2$ :

$$\xi^{(u)}\xi^{(v)}w = \xi^{(u)}\xi^{(v)}u\eta^{(v)}w \stackrel{\xi-act.}{=} \xi^{(u)}\xi^{(v)}u\eta^{(v)}w \stackrel{(3)}{=} \xi^{(u)}\xi^{(v)}u\eta^{(v)}w \stackrel{\xi-act.}{=} \xi^{(u)}\xi^{(v)}w$$

We can prove in a similar way that  $v_1 = v_2$ .

Let us observe that the condition (3) implies  $u_1v_1w_1 = u_2v_2w_2$ . It follows that

$$v_1 = v_2.$$

So,  $\sigma : GxG \rightarrow GxG, \sigma(u, v) = \left( \xi^{(u)}v, u\eta^{(v)} \right)$  is a solution for (1).

□

**Exercise.**(for good students) Using the previous hypotheses prove that  $\sigma : GxG \rightarrow GxG, \sigma(u, v) = \left( \xi^{(u)}v, u\eta^{(v)} \right)$  is an invertible application and find its inverse.

**Remark 3.2** There exists a reciprocal of the previous theorem (see [5]): For a large class of invertible solutions of (1) (i.e. nondegenerate solutions) there exists a group  $G = G(S, \sigma)$  and an embedding  $i : S \rightarrow G$  such that  $\sigma \circ (ixi) = (ixi) \circ \sigma^G$ .

**Commentaries and Conclusions**

P. Etingof, T. Schedler and A. Soloviev gave a complete classification of the non-degenerate solutions for the set-theoretical Yang-Baxter equation (see [1])

In the general case, we consider the vector space  $V$  instead of the set  $S$ . The tensor product  $V \otimes V$  replaces  $S \times S$ .  $R : S \times S \rightarrow S \times S$  is a linear application such that

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}.$$

The problem of finding invertible solutions for (1) is much harder in this case. The classification of all solutions was obtained only for the case  $\dim V = 2$  (using computer calculations).

In the general case, we can produce invertible solutions from quasitriangular Hopf Algebras (there exist some similarities between this method and the construction from section 2 of the current paper). A new method to construct solutions for (1) was introduced in [6].

There are connections of the Yang-Baxter equation with the Inverse Scattering Method and Tzitzeica Surfaces. (The transformations that generate the family of such surfaces found by Tzitzeica in 1910 and their generalizations, are known in the modern literature on integrable equations as Darboux or Bäcklund transformations. They are used to construct the soliton solutions starting from some trivial solutions of the equation  $u_{xy} = e^u - e^{-2u}$ .) For more details see [2].

### References

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