NUMERICAL CHARACTERISTICS OF UNIFORME BIRKHOFF UNIVARIATE INTERPOLATION SCHEMES

by
Nicolae Crainic

Abstract: *In this article we will determine estimative values for the numbers* $a_r(Z) = \left|\{A : s(Z,A) \neq 0\}\right|$ *that correspond to a uniforme Birkhoff univariate interpolation scheme* (Z,S,A), *where* Z *is a set of* n *nodes,* A *is the set of interpolated derivates,* $r = |A|$, *and* $s(Z,A) = \left|\{S : (Z,S,A) \text{ is regular}\}\right|$.

For the beginning we present the following:

1. *The uniform univariate Birkhoff interpolation scheme* is the triplet (Z,S,A) consisting of a set Z of n real numbers, an inferior set $S = \{0,1,...,k\} \subset IN$ which defines the space of the interpolation polynomials

$$P_S = \left\{ P \in IR[x] : P(x) = \sum_{i \in S} a_i x^i, a_i \in IR \right\},$$

and a finite set $A \subset S$, which designates the derivatives with which we interpolate. Regarding all these notions, we can set the associated (uniform, bivariate) Birkhoff *interpolation problem,* which consists in determining the polynomials that satisfy the equations: $\frac{\partial^\alpha P}{\partial x^\alpha}(x) = c_\alpha, (\forall)\alpha \in A, x \in Z,$

where c_α are arbitrary constants.

2. As it follows from the abstract, we denote by $s(Z,A)$ the number of inferior sets S for which the interpolation scheme (Z,S,A) is regular, and by $a_r(Z)$ the number of sets A for which (Z,S,A) is regular for at least one choice of set S.

3. The scheme (Z,S,A) is *normal* if $n|A| = |S|$ (where $|A|$ is the number of the elements of the set A, and $|S|$ is the number of the elements of S), and in this case we write the determinant of the interpolation system as $D(Z,S,A)$.

4. The scheme (Z,S,A) is *regular (singular*) if $D(Z,S,A)$ does not vanish (does vanish) for any choice of the set Z of nodes and is *almost regulate* if $D(Z,S,A)$ is not identical null.

5. The interpolation scheme (Z,S,A) satisfy *the Pólya condition* if

$$a_i \leq n \cdot i, (\forall) 0 \leq i \leq s,$$

where $A = \{a_0, a_1, ..., a_s\}$ with $a_0 < a_1 < ... < a_s$,

6. The Pólya condition is a *sufficient and necessary criterion* for the almost

regularity of an interpolation scheme (Z,S,A).

In case of regularity of an uniform Birkhoff interpolation scheme, the Pólya condition states that the number $a_r(Z)$ is equal to the number $(r-1)$-tuples $(i_1,....i_{r-1})$ that satisfy $i_1 < i_2 < ... < i_{r-1}$, $1 \le i_k \le kn$, for any $1 \le k \le r-1$. This fact shows that the problem of computing the numbers $a_r(Z)$ is combinatorial, and this offers us the possibility of an explicit calculation of these numbers (at least for the first values of r).

1. Proposition. *If* $a_r(Z) = \left|\{A : s(Z,A) \ne 0\}\right|$, *then for any* $n \in IN$ *the following take place:*

$$a_1(Z) = 1,$$
$$a_2(Z) = n,$$
$$a_3(Z) = \frac{n(3n-1)}{2}$$
$$a_4(Z) = \frac{n(2n-1)(4n-1)}{3}$$
$$a_5(Z) = \frac{n(5n-1)(5n-2)(5n-3)}{24}.$$

Proof: The first two formulas are obvious. For the computation of $a_3(Z)$ we must find out the numbers of the non-null natural numbers pairs (i_1,i_2) in the conditions

$$i_1 < i_2,\ i_1 \le n,\ i_2 \le 2n.$$

We have two possibilities. The first one is to have $1 < i_1 < i_2 \le n$, and the number of these pairs coincides with the number of choices of two numbers from a set of n numbers, i.e. C_n^2. The second one is to have $1 < i_1 \le n < i_2 \le 2n$, and the number of these pairs is n^2. Summing up, it follows exactly the formula from the enunciation.

We compute now $a_4(Z)$, i.e. the number of the triplets (i_1,i_2,i_3) with

$$1 \le i_1 < i_2 < i_3,\ i_1 \le n,\ i_2 \le 2n,\ i_3 \le 3n.$$

Again we consider more possible cases (in fact all possibilities):

(1) $1 \le i_1 \le n < i_2 \le 2n < i_3 \le 3n$;

(2) $1 \le i_1 < i_2 \le n < 2n < i_3 \le 3n$;

(3) $1 \le i_1 < i_2 \le n < i_3 \le 2n$;

(4) $1 \le i_1 \le n < i_2 < i_3 \le 2n$;

(5) $1 \le i_1 < i_2 < i_3 \le n$.

The first case produces n^3 possibilities. Each of the cases (2), (3) and (4) produces nC_n^2 possibilities, and the case (5) produces C_n^3 possibilities. It follows a total of

$$n^3 + 3n\frac{n(n-1}{2} + \frac{n(n-1)(n-2)}{6} = \frac{n(2n-1)(4n-1)}{3}$$

possibilities.

We compute now $a_5(Z)$. We have to count those (i_1, i_2, i_3, i_4), satisfying conditions similar to the above ones. We distinguish the following cases:

(1) $1 \le i_1 < i_2 < i_3 < i_4 \le n$. The number of possibilities in this case coincides with the choices of four elements from a set with n elements, i.e. C_n^4.

(2) $1 \le i_1 < i_2 < i_3 \le n < i_4 \le 4n$. In this case we have C_n^3 possible choices for (i_1, i_2, i_3) and $3n$ choices for i_4. Thus the total number of possibilities is $3nC_n^3$.

(3) $1 \le i_1 < i_2 \le n < i_3 < i_4 \le 3n$. In this case we have C_n^2 possible choices for (i_1, i_2) and C_{2n}^2 for (i_3, i_4). Thus the total number of possibilities is $C_n^2 C_{2n}^2$.

(4) $1 \le i_1 < i_2 \le n < i_3 \le 3n < i_4 \le 4n$. In this case we have C_n^2 possible choices for (i_1, i_2), $2n$ choices for i_3, and n possible choices for i_4. Thus the total number of possibilities is $2n^2 C_n^2$.

Analogously we have:

(5) $1 \le i_1 \le n < i_2 < i_3 < i_4 \le 2n$ where the number of possibilities is nC_n^3,

(6) $1 \le i_1 \le n < i_2 < i_3 \le 2n < i_4 \le 4n$ with $2n^2 C_n^2$ possibilities,

(7) $1 \le i_1 \le n < i_2 < i_3 \le 3n < i_4 \le 4n$ with $n^2 C_n^2$ possibilities and

(8) $1 \le i_1 \le n < i_2 \le 2n < i_3 \le 3n < i_4 \le 4n$ with n^4 possibilities.

Summing up, we obtain

$$C_n^4 + 4nC_n^3 + 5n^2 C_n^2 + C_n^2 C_{2n}^2 + n^4 =$$

$$= n\frac{125n^3 - 150n^2 + 55n + 6}{24} =$$

$$= n\frac{(5n-1)(25n^2-25n+6)}{24}$$

possibilities, which is exactly the formula from the enunciation of the proposition. □.

2. Remark. Analyzing the above formulas, we can notice that they contain the same type of expressions, and this fact makes us believe that a simple general formula for all numbers $a_r(Z)$, exists, namely:

$$a_r(Z) = \frac{1}{r} C_m^{r-1}.$$

Also we remark that in order to prove this formula is sufficient to verify it for $r-1$ distinctive values of n. Even more, this formula can be rewritten as a polynomial identity. In order to see this, we remark first that the choice of a $(r-1)$-tuple $(i_1,....i_{r-1})$ as above is the same with the choice of a $(r-1)$-tuple $(j_1,....j_{r-1})$ of natural numbers that satisfy the inequalities:

$$j_1 \geq 1, j_1 + j_2 \geq 2, j_1 + j_2 + j_3 \geq 3, \ ...$$

and whose sum is $r-1$. Depending on the initial $(r-1)$-tuples, j_k is the number of the $i \in \{i_1,....i_{r-1}\}$ elements with $(k-1)n < i \leq kn$. Thus, we can have

$$a_r(Z) = \sum C_n^{j_1}...C_n^{j_{r-1}},$$

where the sum is done after all $(j_1,....j_{r-1})$ as above. Thus the initial formula can be also written as:

$$\sum \frac{n(n-1)...(n-j_1+1)}{j_1!} ... \frac{n(n-1)...(n-j_{r-1}+1)}{j_{r-1}!} = n\frac{(nr-1)(nr-2)...(nr-r+2)}{(r-1)!}$$

.

This equality makes sense for any real number n and it is the equality of two polynomials of degree $(r-1)$. This proves the given statements.

References

1.Crainic, M., Crainic N., Birkhoff interpolation with rectangular sets of nodes, Utrecht University Preprint, nr 1266 January 2003 (trimis spre publicare la Journal of Numerical Analysis).

2.Stancu, D. Dimitrie, Coman Gheorghe, Agratini Octavian, Trâmbiţaş Radu, Analiză numerică şi teoria aproximării, Vol. I, Presa Universitară Clujană, Universitatea "Babeş – Bolyai", 2001.

3.Stancu, D. Dimitrie, Coman Gheorghe, Blaga Petru, Analiză numerică şi teoria aproximării, Vol. II, Presa Universitară Clujană, Universitatea "Babeş – Bolyai", 2002.