ON SOME PROPERTIES OF BRANDT GROUPOIDS

by
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Abstract. In this paper we construct a cohomology theory for Brandt groupoids which extends the usual cohomology theory for groups.

1. INTRODUCTION

We construct a cohomology theory for cochains of a Brandt groupoid *G* with values in an abelian group *A*. This construction was inspired by that of M. Hall Jr. [2] in the case of groups. We also construct a cohomology theory for normalized cochains of a Brandt groupoid *G* with values in an abelian group *A*. In [3] has proved that the cohomology groups for unnormalized cochains are isomorphic to cohomology groups for normalized cochains.

2. PRELIMINARIES

In this section preliminary definitions regarding the category of groupoids and some important examples of groupoids are given.

Definition 2.1 ([1])

A Brandt groupoid is a nonempty set *G* endowed with:

a) two maps *d* and *r* (called source, respectively target), $d, r : G \to G$;

b) a product map

$$m : G_{(2)} \to G,$$

$$(x, y) \to m(x, y) \stackrel{def}{=} xy$$

where $G_{(2)} \stackrel{def}{=} \{ (x, y) \in G \times G \mid d(x) = r(y) \}$ is a subset of $G \times G$ called the **set of composable pairs**;

c) an **inverse map**

$$i : G \to G,$$

$$x \to i(x) \stackrel{def}{=} x^{-1}$$

such that the fallowing conditions are satisfied:

i) $(x, y); (y, z) \in G_{(2)} \Rightarrow (xy, z); (x, yz) \in G_{(2)}$ and $(xy)z = x(yz)$;

ii) $x \in G \Rightarrow (r(x), x); (x, d(x)) \in G_{(2)}$ and $r(x)x = xd(x) = x$;

iii) $x \in G \Rightarrow (x^{-1}, x); (x, x^{-1}) \in G_{(2)}$ and $x^{-1}x = \mathrm{d}(x)$; $xx^{-1} = \mathrm{r}(x)$.

If G is a groupoid then $G_0 \overset{def}{=} d(G) = r(G)$ is the unit set of G and we say that G is a G_0-groupoid.

Example 2.1 The null groupoid.

The set G_0 may be given the following groupoid structure: $d = r =$ the identity map on G_0. u and v are composable if $u = v$ and $u \cdot u = u$.

Example 2.2 The trivial groupoid on X with group Γ.

Let X be a set and Γ be a group having e as unity. We give to $G \overset{def}{=} X \times X \times \Gamma$ the following structure:

$$d(x, y, g) \overset{def}{=} (y, y, e);$$
$$r(x, y, g) \overset{def}{=} (x, x, e).$$

(x, y, g) and (y', z, g') are composable if $y' = y$, $(x, y, g) \cdot (y, z, g') \overset{def}{=} (x, z, gg')$ and $(x, y, g)^{-1} \overset{def}{=} (x, y, g^{-1})$.

Example 2.3 The grouopoid $G^{(n)}$ $(n \geq 2)$.

Let G be a G_0-groupoid and by $G^{(n)}$ we denote the set of n-tuples $(x_0, \ldots, x_{n-1})$ of G such that $(x_{i-1}, x_i) \in G_{(2)}$ for $i = 1, 2, \ldots, n$-1. We give to $G^{(n)}$ the following groupoid structure:

$$d^{(n)}, r^{(n)} : G^{(n)} \to G^{(n)};$$
$$d^{(n)}(x_0, \ldots, x_{n-1}) \overset{def}{=} (x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, d(x_{n-2}x_{n-1}));$$
$$r^{(n)}(x_0, \ldots, x_{n-1}) \overset{def}{=} (x_0, x_1, \ldots, x_{n-2}, r(x_{n-1}));$$

$(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ are composable if $y_0 = x_0x_1$, $y_1 = x_{n-2}x_{n-1}$ and

$$(x_0, \ldots, x_{n-1})(x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, y_{n-1})) \overset{def}{=} (x_0, \ldots, x_{n-2}, x_{n-1}y_{n-1})$$

and the inverse of $(x_0, \ldots, x_{n-1})$ is defined by:

$$(x_0, \ldots, x_{n-1})^{-1} \overset{def}{=} (x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, x_{n-1}^{-1})$$

Definition 2.2

Let G and G' be groupoids.

i) a map f : $G \to G'$ is **a morphism** if for any $(x, y) \in G_{(2)}$ we have $(f(x), f(y)) \in G'_{(2)}$ and $f(x, y) = f(x)f(y)$.
ii) two morphisms $f, g : G \to G'$ are **similar** ($f \sim g$) if there exists a map $\theta : G_0 \to G'$ such that $\theta(r(x))f(x) = g(x)\theta(d(x))$ for any $x \in$ G.

iii) the groupoids G and G' are **similar** ($f \sim g$) if there exists two morphisms $f : G \to G'$ and $g : G' \to G$ such that $g \circ f$ and $f \circ g$ are similar to identity isomorphisms.

Example 2.4

Let $G^{(0)} = G_0$; $G^{(1)} = G$; $G^{(n)}$ be the groupoid given in example 2.3 and $\varphi : G \to G'$ be a morphism of groupoids. The map $\varphi^{(n)} : G^{(n)} \to G'^{(n)}$ defined by:

$$\varphi^{(n)}(x_0, \dots , x_{n-1}) \overset{def}{=} (\varphi(x_0), \dots , \varphi(x_{n-1})) \qquad (1)$$

is a morphism of groupoids for any $n \geq 0$.

Example 2.5

The trivial groupoid $G = X \times X \times \Gamma$ and the group Γ are similar.

3. COHOMOLOGY GROUPS OF A GROUPOID

We assume that:

a) G is a G_0-groupoid;
b) $(A, +)$ is an abelian group;
c) G operates on the left on A

i.e. $G \times A \to A$, $(x, a) \to x.a$ subject of the following conditions:

i) $x.(y.a) = (xy).a$ for all $(x, y) \in G_{(2)}$ and $a \in A$;
ii) $u.a = a$ for all $u \in G_0$ and $a \in A$;
iii) $x.(a + b) = x.a. + x.b$ for all $x \in G$ and $a, b \in A$.

In this hypothesis we say that A is a G-module.

Definition 3.1

Given a G-module A, a function $f : G^{(n)} \to A$, $(x_0, \dots , x_{n-1}) \to f(x_0, \dots , x_{n-1})$, where $G^{(1)} = G$ and $G^{(n)}$ for $n \geq 2$ is the groupoid given in example 1.3, is called a n-cochain of the groupoid G with values in A.

We denote by $C^{(n)}(G, A)$ the additive group of n-cochains of G. By definition $C^{(n)}(G, A) = 0$ if $n < 0$ and $C^{(0)}(G, A) = A$. Define the coboundary operator:

$$\delta^n : C^n(G, A) \to C^{n+1}(G, A)$$

$$\delta^0 f(x) = x.f(d(x)) - f(r(x))$$

for all $x \in G$, $f \in C^0(G, A)$ if $n = 0$;

$$\delta^n f(x_0, \dots , x_n) = x_0.f(x_1, \dots , x_n) + \sum_{i=1}^{n} (-1)^i f(x_0, \dots , x_n) + (-1)^{n+1} f(x_0, \dots , x_{n-1})$$

if $n \geq 1$.

The map $f \to \delta^n f$ is a homomorphism with respect to addition and we have that $(C^n(G, A), \delta^n)$ is a cochain complex. The cohomology n-groups $H^n(G, A)$ of G-module A are defined by $H^n(G, A) = Z^n(G, A) / B^n(G, A)$, where $Z^n(G, A) = \ker \delta^n$ and $B^n(G, A) = \operatorname{Im} \delta^{n-1}$.

If $\varphi : G \to G'$ is a morphism of groupoids then the morphism of groupoids $\varphi^{(n)} : G^{(n)} \to G'^{(n)}$ given by (1) induce a homomorphism of groups defined by:

$$\varphi'^n : C^n(G', A) \to C^n(G, A)$$

$$\varphi'^n(f) \overset{def}{=} f \circ \varphi^{(n)} \text{ for each } f \in C^n(G', A)$$

and satisfying the following:

$$\delta^n \circ \varphi'^n = \varphi'^{n+1} \circ \delta^n \text{ for all } n \geq 0.$$

From here it follows that φ'^n induce a homomorphism of cohomology groups $(\varphi^n)^* : H^n(G', A) \to H^n(G, A)$ given by:

$(\varphi^n)^*([f]) = [\varphi'^n(f)]$ for every $[f] \in H^n(G', A)$.

Remark 3.1

a) $H^0(G, A) = \{ x \in A \mid x.a = a \text{ for all } x \in G \}$ is the set of elements of A such that G operates simply on A. In particular, if G operates trivially on A, i.e. $x.a = a$ for all $x \in G$, then $H^0(G, A) = A$.

b) $Z^1(G, A) = \{ f : G \to A \mid f(x_0x_1) = f(x_0) + x_0f(x_1) \text{ for each } (x_0, x_1) \in G_{(2)}\}$ is the group of **crossed morphisms** of groupoids.

References

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