EXPONENTIAL STABILITY OF STOCHASTIC DISCRETE-TIME, PERIODIC SYSTEMS IN HILBERT SPACES

by
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Abstract. In this paper we consider the linear discrete time systems with periodic coefficients and independent random perturbations (see [4] for the finite dimensional case). We give necessary and sufficient conditions for the exponential stability property of the discussed systems. In order to obtain these characterizations we use either the representations of the solutions of these systems obtained by the authoress in [5] or the Lyapunov equations. These results are the periodic versions of those given in [5].

Key Words: periodic systems, exponential stability, Lyapunov equations.

1. Introduction

In this paper we treat the problem of the exponential and uniform exponential stability of time-varying systems described by linear difference equations, with periodic coefficients, in Hilbert spaces. We yield some characterizations of the uniform exponential stability property, which used the two representation theorems of the solutions of these systems given in [5]. We also prove that in the periodic case (but not in the general case), the uniform exponential stability is equivalent with the exponential stability. Another necessary and sufficient conditions for the exponential stability was obtained in terms of Lyapunov equations.

2.Preliminaries

Let H be a real separable Hilbert space and $L(H)$ be the Banach space of all bounded linear operators transforming H into H. We write $\langle .\,,.\rangle$ for the inner product and $\|.\|$ for norms of elements and operators. We denote by $a \otimes b,\ a,\ b \in H$ the bounded linear operator of $L(H)$ given by $a \otimes b(h) = \langle h, b\rangle a$ for all $h \in H$.

2.1 Nuclear operators

The operator $A \in L(H)$ is said to be nonnegative, and we write $A \geq 0$, if A is self adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We say that $A \in L(H)$ is a positive operator ($A > 0$) if there exists $\gamma > 0$ such that $A > \gamma I$, where I is the identity operator on H. For $A \in L(H)$, $A \geq 0$ we denote by $A^{1/2}$ the square root of A (see [2]) and by $|A|$ the operator $(A^*A)^{1/2}$. Let $A \in L(H)$, $A \geq 0$ and $\{e_n\}_{n \in N^*}$ be an orthonormal basis in H. We define

$Tr(A)$ by $Tr(A)= \sum_{n=1}^{\infty} < Ae_n, e_n >$. It is not difficult to see that $Tr(A)$ is a well defined number independent of the choice of the orthonormal basis $\{e_n\}_{n \in N^*}$.

If $A \in L(H)$ we put $\|A\|_1 = Trace(|A|)_1 = Trace(|A|) \leq \infty$ and we denote by $C_1(H)$ the set $\{A \in L(H) / \|A\|_1 < \infty\}$. The elements of $C_1(H)$ are called nuclear operators.

It is known (see [3]) that $C_1(H)$ (the operators' trace class) is a Banach space endowed with the norm $\|.\|_1$ and for all $A \in L(H)$ and $B \in C_1(H)$ we have AB, $BA \in C_1(H)$.

We denote by H and N the subspaces of L(H) and $C_1(H)$ formed by all self adjoint operators and by K (respectively K_1) the cones of all nonnegative operators of H (respectively N). H is a Banach space and since N is closed in $C_1(H)$ with respect to $\|.\|_1$ we deduce that it is a Banach space, too.

2.2 Covariance operators

Let (Ω, □, P) be a probability space and ξ be a real (or H) valued random variable on Ω. WE write $E(\xi)$ for his mean value (expectation). We denote by $L^2 = L^2(\Omega, □, P, H)$ the space of all equivalence class of H-valued random variables ξ such that $E\|\xi\|^2 < \infty$, (with respect to the equivalence relation $\xi \sim \eta \Leftrightarrow E(\|\xi - \eta\|^2) = 0$).

It is useful to recall (see [1]) that if ξ is a H valued random variable such as $E\|\xi\|^2 < \infty$, then we have $\langle E(\xi), u\rangle = E\langle \xi, u \rangle$ for all $u \in H$.

If $\xi \in L^2$, we define the operator $E(\xi \otimes \xi) : H \to H$, $E(\xi \otimes \xi)(u) = E(\langle u, \xi\rangle \xi)$ for all $u \in H$.

It is easy to see that $E(\xi \otimes \xi)$, which is called the covariance operator of ξ, is a linear, bounded and nonnegative operator. The operator $E(\xi \otimes \xi)$ is nuclear and

$$\| E(\xi \otimes \xi) \|_1 = E\| \xi \|^2. \quad (1)$$

2.3 Representations of the solutions of linear discrete-time systems

Let us consider the stochastic system

$$x_{n+1} = A_n x_n + \xi_n B_n x_n,$$
$$x_k = x \quad (2)$$

where $n, k \in$ N, $n \geq k$, A_n, B_n, $\in L(H)$ and ξ_n are real independent random variables, which satisfy the conditions $E(\xi_n) = 0$ and $E|\xi_n|^2 = b_n < \infty$ for all $n \in$ **N**.

We denote by $X(n, k)$, $n \geq k \geq 0$ the random evolution operator associated with the linear system (2) i.e $X(k, k) = I$ and $X(n, k) = (A_{n-1} + \xi_{n-1} B_{n-1}) \ldots (A_k + \xi_k B_k)$ for all $\dot{n} > k$.

If $x_n = x_n(k, x)$ is the solution of the system (2) then it is unique and $x_n(k, x) = X(n, k)x$.

It is not difficult to see that $E(x_n \otimes x_n)$ is a nuclear , nonnegative operator and

$$\|E(x_n \otimes x_n)\|_1 = E\|x_n\|^2. \quad (3)$$

We consider the linear operator $U_n : N \to N$,

$$U_n(Y) = A_n Y A^*_n + b_n B_n Y B^*_n \tag{4}$$

Which is well-defined because N is a (left and right) ideal of space $L(H)$. Since $\| U_n(Y) \|_1 \leq (\| A_n \|^2 + \| B_n \|^2) \| Y \|_1$ we deduce that $U_n \in L(N)$. We associate to (2) the deterministic system defined on N :

$$y_{n+1} = U_n y_n , \tag{5}$$
$$y_k = R , R \in N'$$

If $Y(n, k)$ is the evolution operator associated with the system (5) then $Y(n, k) = U_{n-1}U_{n-2} \ldots U_k$ if $n-1 \geq k$ and $Y(k, k) = I$, where I is the identity operator on N. Since, $U_n \in L(N)$ it follows that $Y(n, k) \in L(N)$ for all $n \geq k \geq 0$.

Let us denote by $y_n = y_n(k, R)$ the solution of (5) with $y_k = R \in N$; it is clear that it is unique and $y_n(k, R) = Y(n, k)(R)$ for all $n, k \in \mathbf{N}, n \geq k, R \in N$.

The fallowing theorem gives a representation of the covariance operator associated to the solution of (2) by using the evolution operator $Y(n, k)$.

Theorem 1 (see [5])If $x_n = x_n(k, x)$ is the solution of (2), then $E(x_n \otimes x_n)$ is the solution of the system (5) with the initial condition $y_k = x \otimes x$. So

$$E\| X(n, k)x \|^2 = \| Y(n, k) (x \otimes x) \|_1 \tag{6}$$

for all $n \geq k \geq 0$ and $x \in H$.

We consider the mapping $Q_n : H \to H$

$$Q_n(S) = A_n^* S A_n + b_n B_n^* S B_n , \tag{7}$$

where A_n , B_n and $b_n = E | \xi_n |^2 < \infty$ are defined as above.

It is easy to see that Q_n is a linear and bounded operator.

Let us define the operator $T(n, k)$ by $T(n, k) = Q_k Q_{k+1} \ldots Q_{n-1} \in L(H)$ for all $n-1 \geq k$ and $T(k, k) = I$, where I is the identity operator on H .

Theorem 2 [5] If $X(n, k)$ is the random evolution operator associated with the system (2) then we have

$$\langle T(n, k)(S)x, y\rangle = E \langle S X(n, k)x, X(n, k)y\rangle \tag{8}$$

for all $n \geq k \geq 0$, $S \in H$ and $x, y \in H$.

The following lemma is known (see [6]).

Lemma 3 Let $T \in L(H)$. If $T(K) \subset K$ *then* $\| T \| = \| T(I) \|$, where I is the identity operator on H.

Since $Q_p(K) \subset K$ for all $p \in \mathbf{N}$ we deduce that $T(n, k)(K) \subset K$. Then $\| T(n, k) \| = \| T(n, k)(I) \|$.

The following theorem establishes a relation between the operator $T(n, k)$ and the evolution operator $Y(n, k)$.

Theorem 4 [5] If H is a real Hilbert space then

$$\| Y(n, k)(x \otimes x) \|_1 = \langle T(n, k)(I)x, x \rangle$$

and

$$\| T(n, k) \| = \| Y(n, k) \|_1 ,$$

where $\| Y(n, k) \|_1 = \sup_{T \in N, \|T\|_1 = 1} \| Y(n, k)(T) \|_1$ and I is the identity operator on H.

2.4 Periodic solutions of stochastic discrete-time systems

If $\tilde{N} \in \mathbf{N}$, $\tilde{N} > 1$ we say that the sequence $A_n \in L(H)$ (respectively $b_n \in R$) is $\tilde{N}$-periodic if $A_{n+\tilde{N}} = A_n$ (respectively $b_{n+\tilde{N}} = b_n$), $n \geq 0$.

We need the following hypothesis:

H_1 : The sequences A_n, $B_n \in L(H)$ and $b_n = E|\ \xi_n\ |^2$ are $\tilde{N}$-periodic, where A_n, B_n and ξ_n are the coefficients of the system (2). We have the following proposition:

Proposition 5 If H_1 holds and $T(n,k)$, $Y(n,k)$ are the operators introduced in the previous subsection then

a) $T(n + \tilde{N}, k + \tilde{N}) = T(n,k)$ and $T(n\tilde{N}, 0) = T(\tilde{N},0)^n$, $n \geq k \geq 0$,
b) $Y(n + \tilde{N}, k + \tilde{N}) = Y(n,k)$ and $Y(n\tilde{N}, 0) = T(\tilde{N},0)^n$, $n \geq k \geq 0$,
c) $E \| x_{n+\tilde{N}} (k + \tilde{N},x) \|^2 = E \| x_n (k,x) \|^2$ for all $n \geq k \geq 0$.

Proof. Since the operators U_n, Q_n introduced by (4), respectively (7) are Ñ – periodic, the statements a) and b) follows from the definitions of $T(n,k)$ and $Y(n,k)$.

From the relation (6) we obtain

$$E \| x_{n+\tilde{N}} (k+\tilde{N}, x) \|^2 = \| Y (n+\tilde{N}, k+\tilde{N}) (x \otimes x) \|_1$$

and c) is a consequence of b). ■

Remark 6 Assume H_1 holds. From the definition of $T(n, k)$ (respectively $Y(n,k)$) *we deduce that if* $0 \leq r_1, r_2 < \tilde{N}$ and $\alpha \neq \beta$ then

$$T(\alpha\tilde{N} + r_1, \beta\tilde{N} + r_2) = T(\tilde{N}, r_2)\, T(\tilde{N},0)^{\alpha-\beta-1}\, T(r_1, 0)$$

and

$$Y(\alpha\tilde{N} + r_1, \beta\tilde{N} + r_2) = Y(r_1, 0)\, Y(\tilde{N},0)^{\alpha-\beta-1}\, Y(\tilde{N}, r_2).$$

3. Uniform exponential stability

Definition 7 We say that the system (2) is uniformly exponentially stable if there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in \mathbf{N}$ such that we have

$$E\| X(n,k)x \|^2 \leq \beta a^{n-k} \| x \|^2 \tag{9}$$

for all $n \geq k \geq n_0$ and $x \in H$.

Definition 8 The system (2) is exponentially stable if there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in \mathbf{N}$ such that we have

$$E\| X(n,0)x \|^2 \leq \beta a^{n-k} E\| X(k,0)x \|^2 \qquad (10)$$

for all $n \geq k \geq n_0$ and $x \in H$.

From Theorems 2, 9 and the above definitions we obtain the following results.

Theorem 9 The following statements are equivalent:

a) the system (2) is uniformly exponentially stable;

b) there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in$ N such that $\| Y(n,k) \|_1 \leq \beta a^{n-k}$ for all $n \geq k \geq n_0$.;

c) there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in$ N such that $\| T(n,k) \| \leq \beta a^{n-k}$ for all $n \geq k \geq n_0$.

Theorem 10 The system (2) is exponentially stable if and only if there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in$ N such that

$$\langle T(n,0)(I)x, x \rangle \leq \beta a^{n-k} \langle T(k,0)(I)x, x \rangle$$

for all $n \geq k \geq n_0$ and $x \in H$.

Remark 11 If the system (2) is uniformly exponentially stable, then it is exponentially stable.

The converse is not generally true.

Counter - example. Let us consider the system (2), where $B_n = 0$, $A_0 \in L(H)$ is such as $Ker\, A_0 \neq \{0\}$, P_0 is the projection on $Ker\, A_0$ and $A_n = 2P_0$ for all n ≥ 1.

Then for all $n \geq k > 0$ or $n > 1 \geq k$ we get $X(n,0)x = 0$ and (10) holds. If we put $\beta = 2\max\{1, \|A_0\|\}$ and $a = \frac{1}{2}$ then for all $n = k \in \{0,1\}$ or $n = 1, k = 0$ we get (10). Consequently (2) is exponentially stable.

For all $n > k > 1$ we have $X(n,k)x = A_{n-1} \ldots A_k x = 2^{n-k} P_0 x.$

We assume, by contradiction that (2) is uniformly exponentially stable. Then there exist $\beta \geq 1, a \in (0,1)$ such that $\|X(n,k)x\|^2 \leq \beta a^{n-k} \|x\|^2$ for all $n \geq k \geq 0, x \in H$. Thus, for all $n > k > 1, x \in H$ we get $\left\|2^{n-k} P_0 x\right\|^2 \leq \beta a^{n-k} \|x\|^2$ and $\|P_0 x\|^2 \leq \beta \left(\frac{a}{4}\right)^{n-k} \|x\|^2$. As $n - k \to \infty$, we deduce $P_0 = 0$, and we deny the hypothesis. Hence (2) is not uniformly exponentially stable.

The following theorem gives necessary and sufficient conditions for the uniform exponential stability of the system (2), which satisfies the hypothesis H_1 and establishes the equivalence between the exponential stability and the uniform exponential stability.

Theorem 12 The following assertions are equivalent:

a) the system (2) is uniformly exponentially stable<

b) $\lim_{n\to\infty} E\left\|X\left(n\widetilde{N},0\right)x\right\|^2 = 0$ uniformly for $x \in H, \|x\| = 1$;

c) $\rho\left(T\left(n\widetilde{N},0\right)\right) < 1$;

d) $\rho\left(Y\left(n\widetilde{N},0\right)\right) < 1$;

e) the system (2) is exponentially stable.

We denote by $\rho(A)$ the spectral radius of A.

Proof. The implication $a) \Rightarrow b)$ is a consequence of the Definition 7. We will prove $b) \Rightarrow a)$. From Theorem 2 we have $E\left\|X\left(n\widetilde{N},0\right)x\right\|^2 = \left\langle T\left(n\widetilde{N},0\right)(I)x,x\right\rangle$.

Since $\lim_{n\to\infty} E\left\|X\left(n\widetilde{N},0\right)x\right\|^2 = 0$ uniformly for $x \in H, \|x\| = 1$ we deduce that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathrm{N}$ such that $E\left\|X\left(n\widetilde{N},0\right)x\right\|^2 < \varepsilon$ for all $n \geq n(\varepsilon)$ and $x \in H, \|x\| = 1$.

Therefore $\left\langle T\left(n\widetilde{N},0\right)(I)x,x\right\rangle < \varepsilon$ for all $n \geq n(\varepsilon)$ and $x \in H, \|x\| = 1$ or equivalently $\left\|T\left(n\widetilde{N},0\right)(I)\right\| < \varepsilon$ for all $n \geq n(\varepsilon)$.

Let $\varepsilon = \frac{1}{2}$. From Lemma 3 and the last considerations we deduce that there exists $n\left(\frac{1}{2}\right) \in \mathrm{N}$ such as $\left\|T\left(n(\frac{1}{2})\widetilde{N},0\right)\right\| < \frac{1}{2}$. We denote $\hat{N} = n(\frac{1}{2})\widetilde{N}$.

If $n,k \in \mathrm{N}, n \geq k$, then there exist $\alpha,\gamma,r_1,r_2 \in \mathrm{N}, r_1,r_2 < \hat{N}$ such as $n = \alpha\hat{N} + r_1, k = \gamma\hat{N} + r_2$.

If $\alpha \neq \gamma$ we use the Remark 6 and we have $T(n,k) = T(\hat{N},r_2)T(\hat{N},0)^{\alpha-\gamma-1}T(r_1,0)$.

Then

$$\|T(n,k)\| \leq \left\|T(\hat{N},r_2)\right\| \; \left\|T(\hat{N},0)\right\|^{\alpha-\gamma-1}\|T(r_1,0)\|.$$

If we denote $M = \max_{0\leq k\leq n\leq \hat{N}}\|T(n,k)\|$ and $a = (\frac{1}{2})^{\frac{1}{\hat{N}}}$, we obtain $\|T(n,k)\| \leq M^2 a^{n-k} 2^{\frac{\hat{N}+r_1-r_2}{\hat{N}}} \leq 4M^2 a^{n-k}$.

If $\alpha = \gamma$ we have $\|T(n,k)\| \leq \frac{M}{a^{r_1-r_2}} a^{n-k} \leq 2Ma^{n-k}$. Now, we take $\beta = 4M^2 > 2M$ (as $M > 1$) and we deduce that $\|T(n,k)\| \leq \beta a^{n-k}$ for all $n \geq k \geq 0$. The conclusion follows from Theorem 9.

„a) $\Rightarrow$ c)". From T.2.38 of [2] we have

$$\rho(T(\widetilde{N},0)) = \lim_{n\to\infty} \sqrt[n]{\left\| T(\widetilde{N},0)^n \right\|} = \lim_{n\to\infty} \sqrt[n]{\left\| T(n\widetilde{N},0) \right\|}$$

If n_0 is as in the Definition 7 then $\left\| T(n\widetilde{N},0) \right\| = \left\| T(n\widetilde{N} + n_0\widetilde{N}, n_0\widetilde{N}) \right\| \le \beta a^{n\widetilde{N}}$, (Theorem 9). Thus

$$\lim_{n\to\infty} \sqrt[n]{\left\| T(n\widetilde{N},0) \right\|} \le \lim_{n\to\infty} \sqrt[n]{\beta a^{n\widetilde{N}}} \le a^{\widetilde{N}} < 1,$$

and the conclusion follows.

„c) $\Rightarrow$ b)" Let $\rho(T(\widetilde{N},0)) = \lim_{n\to\infty} \sqrt[n]{\left\| T(\widetilde{N},0)^n \right\|} = s < 1$ and let $\varepsilon > 0$ be such that $s + \varepsilon = \alpha < 1$.

Then, there exist $k_0 \in \mathrm{N}$ such that for all $n \ge k_0$ we have $\left\| T(\widetilde{N},0)^n \right\| \le \alpha^n$ and $\left\| T(n\widetilde{N},0) \right\| \le \alpha^n$ (by Proposition 5). Thus $\lim_{n\to\infty} \left\| T(n\widetilde{N},0) \right\| = 0$ or equivalently $\lim_{n\to\infty} \left\| T(n\widetilde{N},0)(I) \right\| = 0$. Using again Theorem 9 we get the conclusion. Since „b) $\Rightarrow$ a)" we get „c) $\Leftrightarrow$ a)".

„c) $\Leftrightarrow$ d)" From Proposition 5 and Theorem 4 we have

$$\left\| T(\widetilde{N},0)^n \right\| = \left\| T(n\widetilde{N},0) \right\| = \left\| Y(n\widetilde{N},0) \right\|_1 = \left\| Y(\widetilde{N},0)^n \right\|_1 .$$

Since

$$\rho(T(\widetilde{N},0)) = \lim_{n\to\infty} \sqrt[n]{\left\| T(\widetilde{N},0)^n \right\|} = \lim_{n\to\infty} \sqrt[n]{\left\| Y(\widetilde{N},0)^n \right\|_1} = \rho(Y(\widetilde{N},0)),$$

we obtain the conclusion.

Now, we prove the equivalence between the uniform exponential stability and the exponential stability.

The implication "a) $\Rightarrow$ d)" is true (see Remark 11).

We only have to prove „d) $\Rightarrow$ a)". From Theorem 10 we see that there exist $\beta \ge 1, a \in (0,1)$ and $n_0 \in \mathrm{N}$ such that we have $\langle T(n,0)(I)x, x \rangle \le \beta a^{n-k} \langle T(k,0)(I)x, x \rangle$ for all $n \ge k \ge n_0$ and $x \in H$.

By Lemma 3 we get $\left\| T(n,0) \right\| \le \beta a^{n-k} \left\| T(k,0) \right\|$ for all $n \ge k \ge n_0$ and

$$\left\| T(n\widetilde{N},0 \right\| \le \beta^{\widetilde{N}} a^{\widetilde{N}(n-n_0)} \left\| T(n_0,0) \right\|^{\widetilde{N}}$$

for all $n \ge n_0$. Then it is clear that $\lim_{n\to\infty} \sqrt[n]{\left\| T(n\widetilde{N},0) \right\|} \le a^{\widetilde{N}} < 1$ and d) $\Rightarrow$ c). Since c) $\Rightarrow$ a), we obtain the conclusion. The proof is complete.

4. The uniform exponential stability and the Lyapunov equations

On the space H we consider the Lyapunov equation

$$P_n = Q_n(P_{n+1}) + I \quad (11)$$

where Q_n is the linear bounded operator given by (7).

Theorem 13 Assume H_1 holds. The system (2) is uniformly exponentially stable if and only if there exist the positive operators $P_0, P_1, ... P_{\tilde{N}-1}$ such that P_n satisfies the equation (11) for $n = 0,..., \tilde{N}-2$ and

$$P_{\tilde{N}-1} = Q_{\tilde{N}-1}(P_0) + I. \quad (12)$$

Proof. Let us prove the implication „$\Rightarrow$". We consider the linear operator $P_n = \sum_{k=n+1}^{\infty} Q_n ... Q_{k-1}(I) + I = \sum_{k=n}^{\infty} T(k,n)(I)$ which is well-defined as the series $\sum_{k=n}^{\infty} \|T(k,n)\|$ converges in **R** (by the hypotensis). We will prove that P_n is $\tilde{N}$ - periodic.

Indeed

$$P_{n+\tilde{N}} = \sum_{k=n+\tilde{N}}^{\infty} T(k, n+\tilde{N})(I) = \sum_{q=n}^{\infty} T(q+\tilde{N}, n+\tilde{N})(I)$$

and by Proposition 5 we get $P_{n+\tilde{N}} = \sum_{q=n}^{\infty} T(q,n)(I) = P_n$. Since

$$Q_n(P_{n+1}) + I = \sum_{k=n+2}^{\infty} Q_n Q_{n+1} ... Q_{k-1}(I) + Q_n(I) + I = \sum_{k=n+1}^{\infty} Q_n ... Q_{k-1}(I) + I = P_n$$

we deduce that P_n is a solution of (11). Thus P_n satisfy the equation (11) for $n = 0,..., \tilde{N}-2$ and $P_{\tilde{N}-1} = Q_{\tilde{N}-1}(P_0) + I.$ As $T(n,k)(K) \subset K$ we see that $P_n \geq I > 0.$ the proof of this implication is complete.

"$\Leftarrow$" Let $P_n, n = 0,1,..., \tilde{N}-1$ be positive operators such that (11) holds for $n = 0,1,..., \tilde{N}-2$ and (12) fulfill. For all $n \in \mathrm{N}$ there exist unique $\alpha, r_1 \in \mathrm{N}, 0 \leq r_1 < \tilde{N}$ such as $n = \alpha\tilde{N} + r_1$ and we define the sequence $P_n = P_{r_1}$. Then $P_n = Q_n(P_{n+1}) + I$ for all $n \in \mathrm{N}$. Thus $P_n = T(n+1,n)(P_{n+1}) + I$ and

$E\langle P_n X(n,k)x, X(n,k)x\rangle = E\langle T(n+1,n)(P_{n+1}) X(n,k)x, X(n,k)x\rangle + E\|X(n,k)x\|^2$ for all $n \geq k$. From Theorem 2 we obtain

$$E\langle P_n X(n,k)x, X(n,k)x\rangle = \langle T(n,k)T(n+1,n)(P_{n+1})x,x\rangle + E\|X(n,k)x\|^2$$
$$= \langle T(n+1,k)(P_{n+1})x,x\rangle + E\|X(n,k)x\|^2$$
$$= E\langle P_{n+1}X(n+1,k)x, X(n+1,k)x\rangle + E\|X(n,k)x\|^2 .$$

From the hypothesis we deduce that there exist $\gamma,\Gamma>0$ such that for all $n\in \mathrm{N}$

$$\gamma I < P_n < \Gamma I \tag{13}$$

So

$$E\langle P_n X(n,k)x, X(n,k)x\rangle \geq$$
$$E\langle P_{n+1}X(n+1,k)x, X(n+1,k)x\rangle + \tfrac{1}{\Gamma}E\langle P_n X(n,k)x, X(n,k)x\rangle.$$

We have

$$(1-\tfrac{1}{\Gamma})E\langle P_n X(n,k)x, X(n,k)x\rangle \geq E\langle P_{n+1}X(n+1,k)x, X(n+1,k)x\rangle$$

and, by induction,

$$(1-\tfrac{1}{\Gamma})^{n+1-k}\langle P_k x,x\rangle \geq E\langle P_{n+1}X(n+1,k)x, X(n+1,k)x\rangle.$$

From (13) it follows $\gamma E\|X(n+1,k)x\|^2 \leq \Gamma(1-\frac{1}{\Gamma})^{n+1-k}\|x\|^2$. If we take $\beta=\frac{\Gamma}{\gamma}\geq 1, \alpha = 1-\frac{1}{\Gamma}$ and $n_0=0$ we obtain the conclusion. The proof is complete. ■

Proposition 14 If the system (2) is uniformly exponentially stable then the equation (11) has a unique $\widetilde{N}-periodic$ and positive solution.

Proof. Let $R_n, n\in \mathrm{N}$ be another $\widetilde{N}$-periodic and positive $(R_n>0)$ solution of (11). We have $P_n - R_n = Q_n(P_{n+1}-R_{n+1}), n\in \mathrm{N}$ and, by induction, $P_n - R_n = T(n+k,n)(P_{n+k}-R_{n+k})$. If $\Gamma>0$ is such that $P_n, R_n < \Gamma I$ for all $n\in \mathrm{N}$ we get $\|P_n - R_n\| \leq \|T(n+k,n)\| \, \|(P_{n+k}-R_{n+k})\| \leq 2\Gamma\|T(n+k,n)\|$. From the hypotheses and from the Theorem 9 we have $\lim_{k\to\infty}\|T(n+k,n)\| = 0$ for all $n\in\mathrm{N}$. As $k\to\infty$ we obtain $P_n = R_n$ for all $n\in\mathrm{N}$. The proof is complete.

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