THE LATTICE OF PRECOMPACT TOPOLOGIES ON F_2^m

by
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Abstract. Chains of totally bounded topologies on abelian groups were studied in [1], [2], [3]. The problem of description of the lattice of all precompact ring topologies on a ring *R* is connected with the Bohr compactification of the ring (R, $\mathcal{T}_d$), where $\mathcal{T}_d$ is the discrete topology on *R* (see, [1]).
The Bohr compactification of the ring ($\mathbb{Z}$, $\mathcal{T}_d$) is described in [1] (Theorem 6.14).
We will study in this paper the lattice $\mathcal{P}(R)$ of all precompact ring topologies on a ring *R*. We focus our attention on some concrete rings:
i) the ring F_2^m, where *m* is some cardinal;
ii) the ring *P*(*X*) of polynomials with integer coefficients over a set *X*;
iii) the free ring *F*(*X*) with identity generated by a set *X*.

Notation and conventions

All ring are assumed to be associative. Topological rings are not necessarily Hausdorff. We will say that a ring *R* has a finite characteristic if there exists $m\in\mathbb{N}$ such that $mx = 0$ for every $x\in$R. The minimal number m with the given property is called the characteristic of *R*.

Denote for a given ring *R* by $\mathcal{P}(R)$ the lattice of all precompact ring topologies on *R*. A subgroup *H* of a group *G* is called *cofinite* provided there exists a finite subset *F* such that $G=F\cdot H$. The Bohr compactification of a topological ring (R, $\mathcal{T}$) is denoted by $b(R, \mathcal{T})$ (see, e.g.,[1]).

If α, β are two ordinal numbers, $\alpha < \beta$, then $[\alpha, \beta)=\{\ \gamma \mid \alpha\leq\gamma<\beta\ \}$. For any ordinal number α the symbol $|\alpha|$ stands for the cardinality of α.

1. Preliminaries

Recall the construction of $\mathcal{T}_1\wedge\mathcal{T}_2$ for two elements $\mathcal{T}_1$ and $\mathcal{T}_2$ from $\mathcal{P}(R)$. Consider the family $\mathcal{B} = \{U+V|U$ is a neighbourhood of zero of $(R, \mathcal{T}_1)$ and V is a neighbourhood of zero of $(R, \mathcal{T}_2)\}$. Then $\mathcal{B}$ gives a precompact ring topology on *R*. This topology is the infimum of $\mathcal{T}_1$ and $\mathcal{T}_1$ in *P*(*R*). The topology $\mathcal{T}_1\wedge\mathcal{T}_2$ is constructed by taking the family $\{U\cap V \mid U$ is a neighbourhood of zero of $(R, \mathcal{T}_1)$, V is a neighbourhood of zero of $(R, \mathcal{T}_2)\}$ as a system of neighbourhoods of zero of (R, $\mathcal{T}_1\wedge\mathcal{T}_2$). We note that the notion of supremum can be given for any family of ring topologies on *R*.

2. Results

Remark 1. a) If *R* is a ring with identity on a ring of finite characteristic and T a precompact ring topology on *R* then (*R*, T) has a fundamental system of neighbourhoods of zero consisting of cofinite ideals.

b) If X is a finite set then for each precompact ring topology T on *P*(*X*) (*F*(*X*)) the topological ring (*P*(*X*), T) ((*F*(*X*), T_1)) is second metrizable.

Consider the ring (P(X), T) and let I={0}. The quotient ring P(X)/I is noetherian and totally bounded. Evidently, P(X) has $\leq\aleph_0$ ideals. Since P(X)/I has a fundamental system of zero consisting of ideals and is countable, it is second countable. Then the ring P(X) is second countable too.

Consider now the ring (F(X), T) and put again I={0}. The quotient ring F(X)/I has a fundamental system of zero consisting of cofinite ideals. Since F(X)/I is finitely generated, it has $\leq\aleph_0$ cofinite ideals (see, [6]). herefore (F(X), T) is a second countable topological ring.

c) A ring R of finite characteristic has the property that every precompact topology on it is metrizable $\Leftrightarrow$ the cardinality of the set of all cofinite ideals of R is $\leq \aleph_0$. Examples of rings with this property are: i) countable noetherian rings with identity; ii) finitely generated rings with identity.

Lemma 1. Let F_2 be the field Z/(2) and R = F_2^X, where X is dome set. Then there exists a bijection between the set of all maximal ideals of R and the set of all ultrafilters on X.

Proof. For any x = $\{x_i\}\in$R, denote Z(x) = $\{i \in X \mid x_i = 0\}$. Denote by A the set of all maximal ideals of R and by B the set of all ultrafilters on X.

Define the mappings α : A $\to$ B and β : B $\to$ A as follows: if I$\in$ A then put α(I) = {Z(i) | i $\in$ I} and if F$\in$ B put β(F) = {x | x $\in$ R, Z(x) $\in$F}.

Claim 1: α(I) $\in$ B.

1) $\Phi\notin\alpha$(I). Indeed, on the contrary there exists i$\in$I such that Z(i) = Φ. Therefore i =1$\in$I. Contradiction.

2) Let Z(x) $\subseteq$A, x $\in$ I. Put y = $\{y_i\}$, $y_i=\begin{cases}0, & \text{if } i\in A\\ 1, & \text{if } i\notin A\end{cases}$

Then yx $\in$ I and Z(yx) = A: if $(yx)_i = 0 = y_i x_i$, then iA. Indeed, on the contrary, i $\notin$ Z(x). Therefore $x_i = 1$ and $y_i = 1$, contradiction. Therefore Z(yx) $\subseteq$A.

If i $\in$A, then $y_i = 0$, therefore $(yx)_i = y_i x_i = 0$ and so i $\in$ Z(yx). We proved that Z(yx)=A.

3) If x,y $\in$ I then Z(x)$\cap$Z(y) = Z(x + y + xy).

Indeed, if $i \in Z(x) \cap Z(y)$ then $x_i = y_i = 0$, and so $x_i + y_i + x_i y_i = 0$, hence $Z(x) \cap Z(y) \subseteq Z(x + y + xy)$. Conversely, let $i \in X$ and $x_i + y_i + x_i y_i = 0$, then $x_i = y_i = 0$; therefore $i \in Z(x) \cap Z(y)$.

Now we will show that $\alpha(I)$ is ultrafilter. Indeed, let $A \cup B = Z(x) \in \alpha(I)$. Put $y \in R$, $y_i = \begin{cases} 0, & \text{if } i \in A \\ 1, & \text{if } i \notin A \end{cases}$, $t_i = \begin{cases} 0, & \text{if } i \in B \\ 1, & \text{if } i \notin B \end{cases}$ and consider $y = (y_i)$, $t = (t_i)$.

Then $yt = x \in I$. By the maximallity of I, then $y \in I$ or $t \in I$. If $y \in I$, $A = Z(y) \in \alpha(I)$; if $t \in I$, $B = Z(t) \in \alpha(I)$. We proved that $\alpha(I)$ is an ultrafilter.

Claim 2: $\beta(\mathsf{F}) \in \mathsf{A}$.

$0 \in \beta(\mathsf{F})$ since $Z(0) = X \in \mathsf{F}$. Let $x,y \in \beta(\mathsf{F}) \Rightarrow Z(x), Z(y) \in \mathsf{F}$. But $Z(x+y) \supseteq Z(x) \cap Z(y) \Rightarrow Z(x+y) \in \mathsf{F} \Rightarrow x+y \in \beta(\mathsf{F})$. Let $x \in \beta(\mathsf{F})$, $y \in R$, then $Z(yx) \supseteq Z(x)$. Therefore $xz \in \beta(\mathsf{F})$. Let $xy \in \beta(\mathsf{F})$; $Z(xy) = Z(x) \cup Z(y) \in \mathsf{F} \Rightarrow Z(x) \in \mathsf{F}$ or $Z(y) \in \mathsf{F}$. Equivalently, $x \in \beta(\mathsf{F})$ or $y \in \beta(\mathsf{F})$.

Claim 3: $\beta\alpha(I) = I$, for every $I \in \mathsf{A}$.

"$\subseteq$" Let $x \in \beta\alpha(I) \Rightarrow Z(x) \in \alpha(I) \Rightarrow \exists y \in I : Z(x) = Z(y) \Rightarrow x = y \in I$.

"$\supseteq$" Let $x \in I \Rightarrow Z(x) \in \alpha(I) \Rightarrow x \in \beta\alpha(I)$.

Claim 4: For every $\mathsf{F} \in \mathsf{B}$, $\alpha\beta(\mathsf{F}) = \mathsf{F}$.

"$\subseteq$" Let $F \in \alpha\beta(\mathsf{F})$, $F = Z(x)$, where $x \in \beta(\mathsf{F}) \Rightarrow Z(x) \in \mathsf{F} \Rightarrow F \in \mathsf{F}$.

"$\supseteq$" Let $F \in \mathsf{F}$. Consider $x \in R$, $F = Z(x) \Rightarrow x \in \beta(\mathsf{F}) \Rightarrow Z(x) \in \alpha\beta(\mathsf{F}) \Rightarrow F \in \alpha\beta(\mathsf{F})$.

Corollary 2. The ring F_2^X has $2^{2^{|X|}}$ different maximal ideals.

Proof. The Stone-Čech compactification βX has the cardinality $|\beta X| = 2^{2^{|X|}}$. By Lemma 1 the cardinality of the set of maximal ideals of R is $|\beta X| = 2^{2^{|X|}}$.

Lemma 3. Let $R = \prod_{\alpha \in \Omega} R_\alpha$ be the topological product of rings R_α where $R_\alpha = \mathsf{Z}/(2)$ ($= \mathsf{F}_2$). Then the lattice of all closed ideals of R is isomorphic to $\mathsf{P}(\Omega)$.

Proof. By Theorem of Numacura [1, p. 30, Th. 3.4] each closed ideal of R has the form $I(\Omega_0)$ where $\Omega_0 \subseteq \Omega$ and $I(\Omega_0) = \{\{x_\alpha\} \in R \mid x_\alpha = 0 \text{ if } \alpha \in \Omega_0\}$. Define the mapping $P(\Omega) \to L$, $\Omega_0 \to I(\Omega_0)$ for each subset $\Omega_0 \subseteq \Omega$.

Claim 1.

1. $I(\Omega_1 \cap \Omega_2) = I(\Omega_1) + I(\Omega_2)$ ($= I(\Omega_1) \vee I(\Omega_2)$).

"$\subseteq$": Let $\{x_\alpha\} \in I(\Omega_1 \cap \Omega_2)$. We will present $\{x_\alpha\}$ as a sum of two elements $\{x_\alpha'\} \in I(\Omega_1)$, $\{x_\alpha''\} \in I(\Omega_2)$.

Case I. $\alpha \in \Omega_1 \backslash \Omega_2$; put $x_\alpha' = 0$, $x_\alpha'' = x_\alpha$.

Case II. $\alpha \in \Omega_1 \cap \Omega_2$; put $x_\alpha' = x_\alpha'' = 0$.

Case III. $\alpha \notin \Omega_2 \backslash \Omega_1$; put $x_\alpha' = x_\alpha$, $x_\alpha'' = 0$.

Case IV. $\alpha \notin \Omega_1 \cup \Omega_2$; put $x_\alpha' = x_\alpha$, $x_\alpha'' = 0$.

By construction, $\{x_\alpha\} = \{x_\alpha'\} + \{x_\alpha''\}$, and $\{x_\alpha'\} \in I(\Omega_1)$, $\{x_\alpha''\} \in I(\Omega_2)$. Therefore $I(\Omega_1 \cap \Omega_2) \subseteq I(\Omega_1) + I(\Omega_2)$. The reverse inclusion is evidenly.

Claim 2.

$I(\Omega_1 \cup \Omega_2) = I(\Omega_1) \cap I(\Omega_2)$ ($= I(\Omega_1) \wedge I(\Omega_2)$).

$\{x_\alpha\} \in I(\Omega_1 \cup \Omega_2) \Leftrightarrow x_\alpha = 0$ for $\alpha \in \Omega_1 \cup \Omega_2 \Leftrightarrow \{x_\alpha\} \in I(\Omega_1)$, $\{x_\alpha\} \in I(\Omega_2)$ $\Leftrightarrow \{x_\alpha\} \in I(\Omega_1) \cap I(\Omega_2)$.

Theorem 4. Let X be an arbitrary infinite set and R = F_2^X. The lattice P(R) is isomorphic to the lattice P (exp exp X).

Proof. By Theorem 1, P(R) is antiisomorphic to the lattice C(bR) of all closed ideals of bR.

We will calculate the ring $b(R, T_d)$ (we note here that the unique invariant of the ring $b(R, T_d)$ is its weight).

By Lemma 2 the cardinality of the set of all maximal ideals of R is card(exp exp X).

One fundamental system B′ of neighbourhoods of zero consists of finite intersection of elements of B. Evidently, $|B'| = 2^{2^{|X|}}$.

By Theorem of Kaplansky [7], $b(R, T_d) \cong \prod_{\alpha \in \Omega} R_\alpha$, where $|\Omega| = 2^{2^{|X|}}$. By Lemma 3 the lattice of closed ideals of $b(R, T_d)$ is antiisomorphic to P(exp exp X).

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