ON A SPECIAL DIFFERENTIAL INEQUALITY

by
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Abstract. We find conditions on the complex-valued functions *A, B, C* defined in the unit disc *U* and the real constants $\alpha, \beta, \gamma, \delta$ such that the differential inequality

$$\operatorname{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3-\beta(zp'(z))^2+\gamma zp'(z)+\delta\right]>0,$$

implies $\operatorname{Re}\, p(z)>0$, where $p\in H\,[1,n]$.

AMS Classification Numbers: 30C80

Keywords: Differential subordination, dominant.

1. Introduction and preliminaries

We let *H*[U] denote the class of holomorphic functions in the unit disc

$$U = \{z \in C : |z| < 1\}.$$

For $a \in \mathbf{C}$ and $n \in \mathbf{N}^*$ we let

$$H\,[a,n] = \{f\in H\,[U],\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$$

and

$$A_n = \{f\in H\,[U],\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots,\ z \in U\}$$

with $A_1 = A$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorema 2.3.i [1, p. 35].

Lemma A. [1, p. 35] *Let* $\psi : \mathbf{C}^2\times U \to \mathbf{C}$ *a function which satisfies*

$$\operatorname{Re}\,\psi(\rho i,\ \sigma,\ z) \le 0,$$

where $\rho,\sigma\in\mathbf{R}$, $\sigma\le -\frac{n}{2}(1+\rho)^2$, $z\in U$ *and* $n\ge 1$.

If $p\in H\,[1,n]$ *and*

$$\operatorname{Re}\,\psi(p(z),\ zp'(z);\ z) > 0$$

then

$$\operatorname{Re}\, p(z) > 0.$$

2. Main results

Theorem. *Let* $\alpha\ge 0,\ \beta\ge 0,\ \gamma\ge 0,\ \delta\le \frac{\alpha n^3}{8}+\frac{\beta n^2}{4}+\frac{\gamma n}{2}$ *and n be a positive integer. Suppose that the function A, B, C :* $U\to\mathbf{C}$ *satisfies:*

i) Re $A(z) \le 0$

ii) Re C $(z) \ge 0$

(1)

iii) $\text{Im}^2 \text{B}(z) \le 4\left[\frac{3\alpha n^3}{8}+\frac{\beta n^2}{4}-\text{Re}\,A(z)\right]\left[\text{Re}\,C(z)+\frac{3\alpha n^3}{8}+\frac{\beta n^2}{2}+\frac{\gamma n}{2}\right].$

If $p \in H[1,n]$ *and*

$$\text{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3-\right.$$
$$\left.-\beta(zp'(z))^2+\gamma zp'(z)+\delta\ \right]>0 \tag{2}$$

then

$$\text{Re}\,p(z) > 0.$$

Proof. We let $\psi: \mathbf{C}^2 \times U \to \mathbf{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3-$$
$$-\beta(zp'(z))^2+\gamma zp'(z) \tag{3}$$

From (2) we have

$$\text{Re}\ \psi(p(z), zp'(z); z) > 0, \text{ for } z \in U. \tag{4}$$

For $\sigma, \rho \in \mathbf{R}$ satisfying $\sigma \le -\frac{n}{2}(1+\rho)^2$, and $z \in U$, by using (1), we obtain:

$$\text{Re}\ \psi(\rho i, \sigma, z) =$$

$$\text{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3-\beta(zp'(z))^2+\gamma zp'(z)+\delta\right]=$$

$$=\text{Re}\left[A(z)p^4 - p^3 iB(z) - p^2 C(z) + \alpha\sigma^3 - \beta\sigma^2 + \gamma\sigma + \delta\right]=$$

$$= p^4\,\text{Re}A(z) - p^3\ Im\ B(z) - p^2\,\text{Re}C(z) + \alpha\sigma^3 - \beta\sigma^2 + \gamma\sigma + \delta \le$$

$$\leq p^4 \operatorname{Re} A(z) - p^3 \operatorname{Im} B(z) - p^2 \operatorname{Re} C(z) - \frac{\alpha n^3}{8}\left(1 + 3p^2 + 3p^4 + p^6\right) -$$

$$-\frac{\beta n^2}{4}\left(1 + 2p^2 + p^4\right) - \frac{\gamma n}{2}\left(1 + p^2\right) + \delta \leq$$

$$\leq -\frac{\alpha n^3}{8} p^6 - p^4\left[\frac{3\alpha n^3}{8} + \frac{\beta n^2}{4} - \operatorname{Re} A(z)\right] + p^3 \operatorname{Im} B(z) -$$

$$-p^2\left[\operatorname{Re} C(z) + \frac{3\alpha n^3}{8} + \frac{\beta n^2}{4} + \frac{\gamma n}{2}\right] - \frac{\alpha n^3}{8} - \frac{\beta n^2}{4} - \frac{\gamma n}{2} + \delta \leq$$

$$\leq -\frac{\alpha n^3}{8} p^6 - p^2\left[p^2\left(\frac{3\alpha n^3}{8} + \frac{\beta n^2}{4} - \operatorname{Re} A(z)\right) - p \operatorname{Im} B(z) +\right.$$

$$\left. + \operatorname{Re} C(z) + \frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2}\right] - \frac{\alpha n^3}{8} - \frac{\beta n^2}{4} - \frac{\gamma n}{2} + \delta \leq 0$$

By using Lemma A we have that Re $p(z) > 0$.

If $\delta = \frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2}$, then theorem can be rewritten as follows:

Corollary 1. *Let $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$, and n be a positive integer. Suppose that the function A, B, C : $U \to \mathbf{C}$ satisfies:*

i) Re $A(z) \leq 0$

ii) Re C $(z) \geq 0$

iii) $\operatorname{Im}^2 B(z) \leq 4\left[\frac{3\alpha n^3}{8} + \frac{\beta n^2}{4} - \operatorname{Re} A(z)\right]\left[\operatorname{Re} C(z) + \frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2}\right]$.

If $p \in H[1,n]$ *and*

$$\operatorname{Re}\left[A(z)p^4(z) + B(z)p^3(z) + C(z)p^2(z) + \alpha(zp'(z))^3 - \beta(zp'(z))^2 +\right.$$

$$\left. + \gamma zp'(z) + \delta + \frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2}\right] > 0$$

then

$$\operatorname{Re} p(z) > 0.$$

If

$A(z) = -1 - 3i, B(z) = 1 - z, C(z) = 5 - 4i, n = 1, \alpha = 2, \beta = 5, \gamma = 3, \delta = 3,$

then in this case from Corollary 1 we deduce:

Example 1. If $p \in H[1,1]$ and

$$\mathrm{Re}\left[(-1-3i)p^4(z)+(1-z)p^3(z)+(5-4i)p^2(z)+2(zp'(z))^3-5(zp'(z))^2+3zp'(z)+3\right]>0$$

then

$$\mathrm{Re}\, p(z) > 0.$$

If $\alpha \equiv 0$, then theorem can be rewritten as follows:

Corollary 2. *Let* $\beta \geq 0, \gamma \geq 0, \delta \leq \frac{\beta n^2}{4}+\frac{\gamma n}{2}$ *and n be a positive integer. Suppose that the function A, B, C :* $U \to \mathbf{C}$ *satisfies:*

i) $\mathrm{Re}\, A(z) \leq 0$

ii) $\mathrm{Re}\, C(z) \geq 0$

iii) $\mathrm{Im}^2\, B(z) \leq 4\left[\frac{\beta n^2}{4}-\mathrm{Re}\, A(z)\right]\left[\mathrm{Re}\, C(z)+\frac{\beta n^2}{2}+\frac{\gamma n}{2}\right]$.

If $p \in H[1,n]$ *and*

$$\mathrm{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)-\beta(zp'(z))^2+\gamma\, zp'(z)+\delta\right]>0$$

then

$$\mathrm{Re}\, p(z) > 0.$$

If $A(z)=-3+i, B(z)=i(2+z), C(z)=4-i, n=3, \beta=2, \gamma=4, \delta=5$, then in this case from Corollary 2 we deduce:

Example 2. If $p \in H[1,3]$ and

$$\mathrm{Re}\left[(-3+i)p^4(z)+i(2+z)p^3(z)+(4-i)p^2(z)-2(zp'(z))^2+4zp'(z)+5\right]>0$$

then

$$\mathrm{Re}\, p(z) > 0.$$

If $\beta \equiv 0$, then Theorem can be rewritten as follows:

Corollary 3. *Let* $\alpha \geq 0, \gamma \geq 0, \delta \leq \frac{\alpha n^3}{8}+\frac{\gamma n}{2}$ *and n be a positive integer. Suppose that the function A, B, C :* $U \to \mathbf{C}$ *satisfies:*

i) $\mathrm{Re}\, A(z) \leq 0$

ii) $\mathrm{Re}\, C(z) \geq 0$

iii) $\mathrm{Im}^2\, B(z) \leq 4\left[\frac{3\alpha n^3}{8}-\mathrm{Re}\, A(z)\right]\left[\mathrm{Re}\, C(z)+\frac{3\alpha n^3}{8}+\frac{\gamma n}{2}\right]$.

If $p \in H[1,n]$ *and*

$$\operatorname{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3+\gamma\, zp'(z)+\delta\ \right]>0$$

then

$$\operatorname{Re} p(z) > 0.$$

If

$$A(z)=-5+4i,\ B(z)=(1+z),\ C(z)=2+i,\ n=4,\ \beta=\frac{3}{2},\ \gamma=7,\ \delta=\frac{1}{5},$$

then in this case from Corollary 3 we deduce:

Example 3. If $p\in H[1,4]$ and

$$\operatorname{Re}\left[(-5+4i)p^4(z)+(1+z)p^3(z)+(2+i)p^2(z)-\frac{3}{2}(zp'(z))^2+7zp'(z)+\frac{1}{3}\right]>0$$

then

$$\operatorname{Re} p(z) > 0.$$

If $\gamma \equiv 0$, then Theorem can be rewritten as follows:

Corollary 4. *Let* $\alpha \ge 0, \beta \ge 0, \delta \le \frac{\alpha n^3}{8}+\frac{\beta n^2}{4}$ *and n be a positive integer. Suppose that the function A, B, C :* $U \to \mathbf{C}$ *satisfies:*

i) $\operatorname{Re} A(z) \le 0$

ii) $\operatorname{Re} C(z) \ge 0$

iii) $\operatorname{Im}^2 B(z) \le 4\left[\frac{3\alpha n^3}{8}+\frac{\beta n^2}{4}-\operatorname{Re} A(z)\right]\left[\operatorname{Re} C(z)+\frac{3\alpha n^3}{8}+\frac{\beta n^2}{2}\right].$

If $p\in H[1,n]$ *and*

$$\operatorname{Re}\left[A(z)p^4(z)+B(z)p^3(z)+C(z)p^2(z)+\alpha(zp'(z))^3-\beta(zp'(z))^2+\delta\ \right]>0$$

then

$$\operatorname{Re} p(z) > 0.$$

If $A(z)=-1+2i, B(z)=+4+z, C(z)=5-i, n=5, \alpha=4, \beta=1, \delta=7$ then in this case from Corollary 4 we deduce:

Example 4. If $p\in H[1,5]$ and

$$\operatorname{Re}\left[(-1+2i)p^4(z)+(4+z)p^3(z)+(5-i)p^2(z)+4(zp'(z))^2+7\right]>0$$

then

$$\operatorname{Re} p(z) > 0.$$

References.

[1] S.S. Miller and P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc. New York, Basel, 2000.

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