A CLASS OF SINGULAR PERTURBATED BILOCAL LINEAR PROBLEMS

by
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Abstract. This paper presents the algorithms for solving a boundary layer bilocal singular perturbated problem when the boundary layer is situated in origin. The solving of this problem has very good results when approximative methods are used. For this problem a uniform first order expansion was obtained.

Key word: boundary layer, uniform expansion, outer and inner expansion, maching.

As it is known from literature there are many methods to determine the uniform solution for the singular perturbated problems attached to same bilocal problems. The study of this kind of singular perturbated problems attached to some ordinary differential equation of high order has good results if we use approximative methods. Thus for these problems we will search asymptotic solutions using the maching asymptotic expansion methods.

Let's consider the following bilocal linear problem:

$$\varepsilon y'' + x^n y' - x^m y = 0 \tag{1a}$$

$$y(0) = \alpha, \quad y(1) = \beta \tag{1b}$$

for which we obtain a first order uniform expansion considering the boundary layer situated in origin.

We suppose that the boundary layer is situated in origin. Thus the outer expansion must satisfy the right limit condition and the inner expansion must satisfy the left limit condition and if there is only one distinct limit then the inner and outer expansion must be mached.

From equation (1a) we can see that the small parameter ε multiply the second order derivative, thus the boundary layer position depend by the coefficient x^n and because the value of y depend by the limits α and β, the position of the boundary layer will depend also by this parameters.

<u>The outer expansion</u>. We choose the first order outer expansion in the form:

$$y^0 = y_0(x) + \varepsilon\, y_1(x) + \varepsilon^2 y_2(x) + \dots \tag{2}$$

and using in equation (1a) and in the rights limit condition, y(1) = β, after equalization of the coefficients of ε^0 we have:

$$x^n y_0{}' - x^m y_0 = 0\,, \qquad y_0(1) = \beta \tag{3}$$

which have the solution:

$$y_0 = \beta x \qquad\qquad if \quad n - m = 1$$

(4a,b)

$$y_0 = \beta \exp\left(\frac{x^{m-n+1} - 1}{m - n + 1}\right) \qquad if \quad n - m \neq 1$$

To determine the inner expansion, we introduce the stretching transformation :

$$\xi = \frac{x}{\varepsilon^{\lambda}}\,, \qquad \lambda > 0$$

in equation (1a) and we obtain

$$\varepsilon^{1-2\lambda}\frac{d^2 y^i}{d\xi^2} + \varepsilon^{(n-1)\lambda}\xi^n \frac{dy^i}{d\xi} - \varepsilon^{m\lambda}\xi^m y^i = 0 \tag{5}$$

With ε → 0, the distinct limits of equation (5) are:

$$\frac{d^2 y^i}{d\xi^2} + \xi^{m+1}\frac{dy^i}{d\xi} - \xi^m y^i = 0\,, \quad if \quad n - m = 0 \quad and \quad m \neq \; -2 \tag{i}$$

corresponding to λ = 1 / (m+2)

$$\frac{d^2 y^i}{d\xi^2} + \xi^n \frac{dy^i}{d\xi} = 0\,, \quad if \quad n - m < 0 \quad and \quad n \neq \; -1 \tag{ii}$$

corresponding to λ = 1 / (n+1), and respectively

$$\frac{d^2 y^i}{d\xi^2} + \xi^m y^i = 0\,, \quad if \quad n - m > 0 \quad and \quad m \neq -2 \tag{iii}$$

corresponding to λ = 1 / (m+2). We can see that the limits are not distinct and the boundary layer in origin don't exist when n = -1 and m = -2, n = -1 and m > -2 or m = -2 and n> -1.

In case (i) the distinct limit is the same like in the original problem and no simplification was made; thus for case n = m+1 it is necessary to solve the original problem.

In case (ii) the general solution of the problem is:

$$y^i = c_1 \int_0^{\xi} \exp\left(-\frac{\tau^{n+1}}{n+1}\right) d\tau + c_2 \tag{6}$$

for n = -1. For n < -1 the integral do to not exist and for $y^{(i)}$ (0) finite $c_1 = 0$. Thus, in the second case, do not exist boundary layer in origin. In the first case, because the inner develop must satisfy the limit condition in origin and for x = 0 correspond to ξ = 0, y^i (0) = α and we obtain $c_2 = \alpha$.

$$y^i = c_1 \int_0^{\xi} \exp\left(-\frac{\tau^{n+1}}{n+1}\right) d\tau + \alpha \tag{7}$$

To obtain c_1 we mach (4b) and (7). We have:

$$\left(y^0\right)^i = \beta \exp\left(-\frac{1}{m-n+1}\right) \tag{8a}$$

$$\left(y^i\right)^0 = c_1 \int_0^{\infty} \exp\left(-\frac{\tau^{n+1}}{n+1}\right) d\tau + \alpha \tag{8b}$$

Taking $\frac{\tau^{n+1}}{n+1} = t$ we have $\tau^n d\tau = dt$, from where we have:

$$\int_0^\infty \exp\left(-\frac{\tau^{n+1}}{n+1}\right)d\tau = \frac{1}{(n+1)^{n/n+1}}\int_0^\infty t^{-n/n+1}e^{-t}dt = (n+1)^{-n/n+1}\Gamma[1/(n+1)]$$

Using (8a) and (8b) we obtain:

$$c_1 = \frac{(n+1)^{n/n+1}}{\Gamma[1/(n+1)]}\left[\beta\exp\left(-\frac{1}{m-n+1}\right)-\alpha\right] \tag{9a}$$

and

$$y^i = \frac{(n+1)^{n/n+1}}{\Gamma[1/(n+1)]}\left[\beta\exp\left(-\frac{1}{m-n+1}\right)-\alpha\right]\int_0^\xi \exp\left(-\frac{\tau^{n+1}}{n+1}\right)d\tau + \tau \tag{9b}$$

Adding (4b) and (9b), and subtracting (8a) we obtain the uniform develop:

$$y^c = \beta\exp\left(\frac{x^{m-n+1}-1}{m-n+1}\right) + \frac{(n+1)^{n/n+1}}{\Gamma[1/(n+1)]}\left[\beta\exp\left(-\frac{1}{m-n+1}\right)-\alpha\right]\int_0^\xi \exp\left(-\frac{\tau^{n+1}}{n+1}\right)d\tau \\ + \alpha - \beta\exp\left(-\frac{1}{m-n+1}\right) \tag{10}$$

in the case n-m <1 and n ≠ -1.

For (iii) the solution of the problem is:

$$y^i = \sqrt{\xi}\left\{c_1 I_\nu\left[\frac{2}{m+2}\xi^{(m+2)/2}\right] + c_2 K_\nu\left[\frac{2}{m+2}\xi^{(m+2)/2}\right]\right\} \tag{10}$$

where

$$\nu = \frac{1}{m+2}, \qquad I_\nu \sim \frac{e^\xi}{\sqrt{2\pi\xi}}, \qquad K_\nu \sim \sqrt{\frac{\pi}{2\xi}}e^{-\xi}, \qquad for \qquad \xi \to 0$$

Maching y^i and y^0 we will obtain $c_1 = 0$. Because the inner develop must satisfy the limit condition in origin and because x = 0 correspond to ξ = 0, $y^i(0) = \alpha$. More, because

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}, \qquad for \qquad z \to 0$$

we have

$$\sqrt{\xi}K_\nu\left[\frac{2}{m+2}\xi^{(m+2)/2}\right] \sim \frac{1}{2}\sqrt{\xi}\Gamma(\nu)\left[\frac{1}{m+2}\xi^{(m+2)/2}\right]^{-\nu} = \frac{1}{2}\left(\frac{1}{m+2}\right)^{-\nu}\Gamma(\nu)$$

Thus imposing the condition that $y^i(0) = \alpha$ we obtain

$$c_2 = \frac{2\alpha}{(m+2)^\nu \Gamma(\nu)}$$

Therefore

$$y^i = \frac{2\alpha}{(m+2)^\nu \Gamma(\nu)}\sqrt{\xi}\, K_\nu\left[\frac{2}{m+2}\xi^{(m+2)/2}\right] \tag{11}$$

Because $(y^0)^i = 0$ and $(y^i)^0 = 0$ it is possible to mach the inner develop and the outer develop. Adding (4b) and (11) we obtain the uniform develop:

$$y^c = \beta \exp\left(\frac{x^{m-n+1}-1}{m-n+1}\right) + \frac{2\alpha}{(m+2)^\nu \Gamma(\nu)}\sqrt{\xi}\, K_\nu\left[\frac{2}{m+2}\xi^{(m+2)/2}\right] \tag{12}$$

in the case (n-m) > 1 and $m \neq -2$.

Thus we determine a first order uniform expansion considering the boundary layer situated in origin.

References

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