THE R - PERFECT MORPHISMES

by
Dumitru Botnaru and Alina Țurcanu

Definition. Let R – a reflective subcategory of the C category. The epimorphisme $p: X \to Y$ is called R- extension if for any object $A \in |R|$, any morphism $f: X \to A$ extends trough the morphisme p, i.e $f = g\,p$ for some morphism p.

We shall denote by γR the class of all R – extensions. The class lower ortogonale morphisms $(\gamma R)^{\perp}$ of the class γR is called the class of R-perfect morphisms. It is well known that in a small cowell category with inductive limits (γR, $(\gamma R)^{\perp}$) is a right-bicategorical structure. For some reflective subcategories the classes γR and $(\gamma R)^{\perp}$ have been described (to see [S]). We shall examine some properties of these classes. For some categories there are conditions when (γR,$(\gamma R)^{\perp}$) is a right-bicategorical structure and there is a process for obtaining (γR, $(\gamma R)^{\perp}$)-factorization of any morphism.

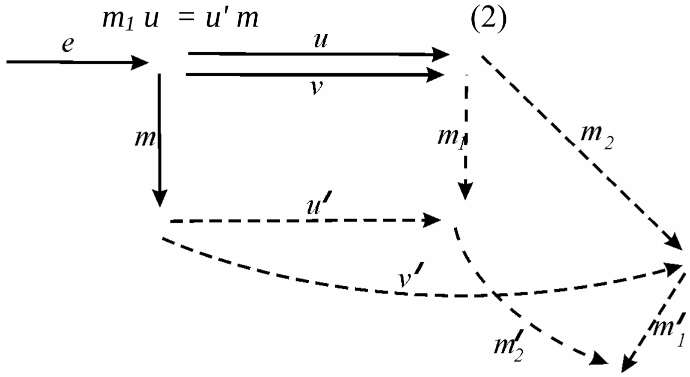
Lemma. Let C – a categorie with puschout square, $m\,e$ - an epi and m - an universal mono. Then e is an epi.

Proof. Let

$$u\,e = v\,e \qquad (1)$$

and we shall prove that $u = v$. We construct the pull-back squares:
on the morphisms m and u

$$m_1\,u = u'\,m \qquad (2)$$



on the morphisms m and v

$$m_2 v = v' m \qquad (3)$$

on the morphisms m_1 and m_2

$$m_2' m_1 = m_1' m_2 \qquad (4)$$

We have

$$m_1' v' me = (\,from(3)) = m_1' m_2 ve = (\,from(1)) = m_1' m_2 ue =$$
$$= (\,from(4)) = m_2' m_1 ue = (\,from(2)) = m_2' u' me$$

i.e.

$$m_1' v' me = m_2' u' me \qquad (5)$$

and since $m\,e$ is epi it follows that

$$m_1' v' = m_2' u' \qquad (6)$$

Then

$$m_2' m_1 u = (\,from(2)) = m_2' u' m = (\,from(6)) = m_1' v' m =$$
$$= (\,from(3)) = m_1' m_2 v = (\,from(4)) = m_2' m_1 v$$

i.e.

$$m_2' m_1 u = m_2' m_1 v \qquad (7)$$

Since m is a universal mono and the squares are (2)-(4) are puschout we deduce that m_1, m_2, m_1' and m_2' are monomorphisms. Thus $m_2' m_1$ is a monomorphisme. Then from the equality (7) it follows that $u = v$.

Corollary. Let C – category with puschout squares, $f\,g$ – an epi and an universal mono. The morphisme g is an epi iff f is an universal mono.

Proof. Let f be an epi. Then the square $f\,g = 1 \cdot (f\,g)$ is puschout and thus f is an universal mono. The reciprocal affirmation follows from the above lemma.

Definition. The class A of morphisms of a category C is called left-stabled if from the fact that $a\,f' = f\,a'$ is a pull-back square and $a \in$ A it follows that $a' \in$ A, too.

Let C be a category with pull-back and puschout squares, the left-stabled class M_u of universal mono, and R be a monoreflective subcategory. Then (γR, (γR $)^{\perp}$) is a right-bicategorical structure. The both classes contain the class of izomorphisms and are closed respect to the composition. It remains to prove that the (γR, (γR $)^{\perp}$) – factorization of the morphisms from the category C . Let $f\colon X \to Y \in$ C, $r^X\colon X \to rX$ and $r^Y\colon Y \to rY$ the R-replicas of the respective objects. Then

$$r^Y f = r(f)\, r^X \qquad (1)$$

for some morphism *r(f)*. We construct the pull-back squares on the morphisms r^Y and *r(f)*:

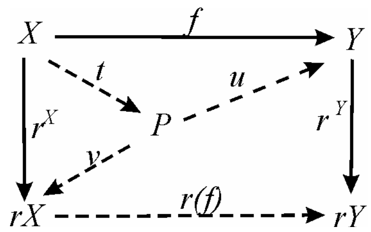
$$r^Y u = r(f)\, v \qquad (2)$$

Then

$$f = u\, t \qquad (3)$$

$$r^X = v\, t \qquad (4)$$

for some morphism *t*. A monoreflective subcategory is at the same time an epireflective and M_u - reflective. Thus, $r^Y \in \mathsf{M}_u$, and by the hypotheses, $v \in \mathsf{M}_u$. By the lemma, from the equality (4) it follows that *t* is an epi. Thus, $t \in \gamma\mathsf{R}$. Further, $\mathsf{R} \subset (\gamma\mathsf{R})^{\perp}$, thus $r(f) \in (\gamma\mathsf{R})^{\perp}$. Since the square (2) is pull-back, we deduce that $u \in (\gamma\mathsf{R})^{\perp}$. . In this way we have proved that the equality (3) is the $(\gamma\mathsf{R}, (\gamma\mathsf{R})^{\perp})$ – factorization of the morphisms *f*. The unity of the factorization follows from the fact that the $\gamma\mathsf{R}$ and $(\gamma\mathsf{R})^{\perp}$ classes are ortogonals. So, I have proved the follow result:



Theorem. Let C be a category with pull-back and puschout squares, the left-stabled class M_u of universal mono, and R be a monoreflective subcategory. Then:

1. $(\gamma\mathsf{R}, (\gamma\mathsf{R})^{\perp})$ is a right-bicategorical structure.
2. For all morphism $f: X \to Y$ the equality $f = u\, t$ is the $(\gamma\mathsf{R}, (\gamma\mathsf{R})^{\perp})$ – factorization of the morphisms *f*.
3. $f \in (\gamma\mathsf{R})^{\perp}$ iff $r(f)\, r^X = r^Y f$ is a pull-back squares.

Remark. We mention that in the catgory of locally convex spaces $\mathsf{C}_2\mathsf{V}$ and $\mathsf{C}_2\mathsf{Ab}$ (of the abeliene local convex groups), the class M_u is left-stable and any non-null reflective subcategory is monoreflective.

References

[1] Strecker G.E., On characterizations of perfect morphisms and epireflective hulls. Lecture Notes in Math., 1974, v.378, p.468-1500.

Authors:

Dumitru Botnaru and Alina Țurcanu - Technical University of Moldova 2005 Bul. Stefan cel Mare, 168, Dep. Math., email:dvbotnaru@mail.md; talina@moldova.cc