A GENERALIZED LAGRANGE IDENTITY AND CAUCHY-BUNIAKOVSKY INEQUALITY

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Abstract. The purpose of this Note is to give an extension to the *Lagrange* identity [1] and [2], and then an extension to the *Cauchy-Bouniakovsky* inequality, since the right hand side is positive. Finally, we give an application to polynomials.

1. Principal result

Let $\mathrm{a}_1,...,\mathrm{a}_s\,;\mathrm{b}_1,...,\mathrm{b}_s \in |$; and

$$M_{2m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\sum a^{m+1-k} b^k\right) \left(\sum a^k b^{m+1-k}\right)$$

where $\sum a^k b^{m+1-k}$ means $\sum_{i=1}^{i=s} a_i^k b_i^{m+1-k}$

Theorem. generalized Lagrange identity

$$\mathrm{M}_{2m} = \begin{cases} \sum\limits_{i<j} (a_i b_j - a_j b_i)^{2r} & \text{for } m = 2r-1 \\ \sum\limits_{i<j} (a_i b_j - a_j b_i)^{2r} (a_i b_j + a_j b_i) & \text{for } m = 2r \end{cases}$$

Corollary. Generalized Cauchy-Buniakovsky inequality

$$\mathrm{M}_{2;2r-1} \geq 0 \;\; \forall a_i ; b_j \in \Re \qquad (\text{for } m = 2r-1)$$

Remark. For $m = 2r$, we deduce the sign of $\mathrm{M}_{2;m}$

If $a_i b_j \geq i \,\&\, j$ then $\mathrm{M}_{2;m} \geq 0$:

If $a_i b_j \leq i \,\&\, j$ then $\mathrm{M}_{2;m} \leq 0$:

Pour $m = 1;2;3;4;...$ on a:

$$\mathrm{M}_{2;1} = \left(\sum a_i^2\right)\left(\sum b_i^2\right) - \left(\sum a_i b_i\right)^2 \quad \rightarrow \quad \text{Lagrange Identity:}$$

$$\mathrm{M}_{2;2} = \left(\sum a_i^3\right)\left(\sum b_i^3\right) - \left(\sum a_i^2 b_i\right)\left(\sum a_i b_i^2\right)$$

$$M_{2;3}=\left(\sum a_i^4\right)\left(\sum b_i^4\right)+3\left(\sum a_i^2b_i^2\right)^2-4\left(\sum a_i^3b_i\right)\left(\sum a_ib_i^3\right)$$
$$M_{2;4}=\left(\sum a_i^5\right)\left(\sum b_i^5\right)+2\left(\sum a_i^3b_i^2\right)\left(\sum a_i^2b_i^3\right)-3\left(\sum a_i^4b_i\right)\left(\sum a_i\,b_i^4\right)$$
$$\vdots$$

2. Proof of the theorem

We have

$$M_{2;m}=\sum a^{m+1}\sum b^{m+1}-\binom{m}{1}\sum a^mb\sum ab^m+\cdots+(-1)^m\binom{m}{m}\sum ab^m\sum a^mb$$

1) if $m=2r-1$

$$M_{2;2-r}=\sum a^{2r}\sum b^{2r}+\sum_{k=1}^{r-1}(-1)^k\binom{2r-1}{k}\left(\sum a^{2r-k}b^k\right)\left(\sum a^kb^{2r-k}\right)+$$
$$+(-1)^r\binom{2r-1}{k}\left(\sum a^rb^r\right)^2+\sum_{k=r+1}^{2r-1}(-1)^k\binom{2r-1}{k}\left(\sum a^{2r-k}b^k\right)\left(\sum a^kb^{2r-k}\right)$$
$$=\sum a^{2r}\sum b^{2r}+\sum_{k=1}^{r-1}(-1)^k\binom{2r-1}{k}\left(\sum a^{2r-k}b^k\right)\left(\sum a^kb^{2r-k}\right)+$$
$$+(-1)^r\binom{2r-1}{k}\left(\sum a^rb^r\right)^2+\sum_{l=1}^{r-1}(-1)^l\binom{2r-1}{2r-l}\left(\sum a^{2r-l}b^l\right)\left(\sum a^lb^{2r-l}\right)$$
$$=\sum a^{2r}\sum b^{2r}+\left(\sum_{k=1}^{r-1}(-1)^k\left\{\binom{2r-1}{k}+\binom{2r-1}{2r-k}\right)\left(\sum a^{2r-k}b^k\right)\left(\sum a^kb^{2r-k}\right)\right)+$$
$$+(-1)^k\binom{2r-1}{r}\left(\sum a^rb^r\right)^2$$
$$=\sum a^{2r}\sum b^{2r}+\left(\sum_{k=1}^{r-1}(-1)^k\binom{2r}{k}\left(\sum a^{2r-k}b^k\right)\left(\sum a^kb^{2r-k}\right)\right)+\frac{1}{2}(-1)^k\binom{2r}{k}\left(\sum a^rb^r\right)^2$$
$$=\sum_{i;j}\left\{\left[\sum_{k=0}^{r-1}(-1)^k\binom{2r}{k}(a_ib_j)^{2r-k}(a_jb_i)^k\right]+\frac{1}{2}(-1)^r\binom{2r}{k}(a_ib_j)^r(a_jb_i)^r\right\}$$

the permutation of i & j gives:

$$= \sum_{i<j}\left\{\left[\sum_{k=0}^{r-1}\binom{2r}{k}(a_i b_j)^{2r-k}(-a_j b_i)^k + \sum_{k=0}^{r-1}\binom{2r}{k}(a_j b_i)^{2r-k}(-a_i b_j)^k\right]+\right.$$

$$\left.+\left[\frac{1}{2}\binom{2r}{k}(a_i b_j)^r(-a_j b_i)^r + \binom{2r}{k}(a_i b_j)^r(-a_j b_i)^r\right]\right\}$$

$$= \sum_{i<j}\left\{\left[\sum_{k=0}^{r-1}\binom{2r}{k}(a_i b_j)^{2r-k}(-a_j b_i)^k + \frac{1}{2}\binom{2r}{r}(a_i b_j)^r(-a_j b_i)^r\right]+\right.$$

$$\left.+\sum_{l=r+1}^{2r}\binom{2r}{l}(a_i b_j)^l(-a_i b_j)^{2r-1}\right\}$$

$$= \sum_{i<j}(a_i b_j - a_j b_i)^{2r}$$

2) if $m = 2r$

$$\mathrm{M}_{2;2r} = \sum a^{2r+1}\sum b^{2r+1} + \sum_{k=1}^{r}(-1)^k\binom{2r}{k}\left(\sum a^{2r-k+1}b^k\right)\left(\sum a^k b^{2r-k+1}\right)+$$

$$+\sum_{k=r+1}^{2r}(-1)^k\binom{2r-1}{k}\left(\sum a^{2r-k+1}b^k\right)\left(\sum a^k b^{2r-k+1}\right)$$

$$= \sum a^{2r+1}\sum b^{2r+1} + \sum_{k=1}^{r}(-1)^k\binom{2r}{k}\left(\sum a^{2r-k+1}b^k\right)\left(\sum a^k b^{2r-k+1}\right)+$$

$$+\sum_{k=r+1}^{r}(-1)^{l+1}\binom{2r}{2r-l+1}\left(\sum a^{2r-l+1}b^l\right)\left(\sum a^l b^{2r-l+1}\right)$$

$$= \sum a^{2r+1}\sum b^{2r+1} + \sum_{k=1}^{r}(-1)^k\left\{\binom{2r}{k}-\binom{2r}{2r-l+1}\right\}\left(\sum a^{2r-k+1}b^k\right)\left(\sum a^k b^{2r-k+1}\right)$$

We remark that $\binom{2r}{k}-\binom{2r}{2r-l+1}=\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}$; witch gives:

$$\mathrm{M}_{2;2r} = \sum_{k=0}^{r}(-1)^k\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}\left(\sum a^{2r-k+1}b^k\right)\left(\sum a^k b^{2r-k+1}\right)$$

$$= \sum_{i;j}\left\{\sum_{k=0}^{r}(-1)^k\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}(a_i b_j)^{2r-k+1}(a_j b_i)^k\right\}$$

The transposition of i& j; gives:

$$\mathrm{M}_{2;2r} = \sum_{i<j}\left\{\sum_{k=0}^{r}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}\left(a_i b_j\right)^{2r-k+1}\left(-a_j b_i\right)^k + \right.$$

$$\left. + \sum_{k=0}^{r}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}\left(a_j b_i\right)^{2r-k+1}\left(-a_i b_j\right)^k\right\}$$

$$\mathrm{M}_{2;2r} = \sum_{i<j}\left\{\sum_{k=0}^{r}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}\left(a_i b_j\right)^{2r-k+1}\left(-a_j b_i\right)^k + \right.$$

$$\left. + \sum_{l=r+1}^{2r}\left(-\frac{2(r-l)+1}{2r+1}\right)\binom{2r+1}{l}\left(a_j b_i\right)^{l}\left(-a_i b_j\right)^{2r-l+1}\right\}$$

$$= \sum_{i<j}\left\{\sum_{k=0}^{2r+1}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}\left(a_i b_j\right)^{2r-k+1}\left(-a_j b_i\right)^k\right\}$$

let $p = a_j b_i$ & $q = -a_i b_j$, we have:

$$\mathrm{M}_{2;2r} = \sum_{i<j}\left\{\sum_{k=0}^{2r+1}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}p^{2r-k+1}q^k\right\}$$

$$= \sum_{i<j}\left\{\sum_{k=0}^{2r+1}\frac{2(r-k)+1}{2r+1}\binom{2r+1}{k}p^{2r-k+1}q^k - \frac{2}{r+1}\sum_{k=1}^{2r+1}k\binom{2r+1}{k}p^{2r-k+1}q^k\right\}$$

$$= \sum_{i<j}\left\{\sum_{k=0}^{2r+1}\binom{2r+1}{k}p^{2r-k+1}q^k - \frac{2(2r+1)}{r+1}\sum_{k=1}^{2r+1}k\binom{2r}{k-1}p^{2r-k+1}q^k\right\}$$

$$= \sum_{i<j}\left\{(p+q)^{2r+1} - 2\sum_{l=1}^{2r}k\binom{2r}{l}p^{2r-l}q^{l+1}\right\}$$

$$= \sum_{i<j}\left\{(p+q)^{2r+1} - 2q(p+q)^{2r}\right\}$$

$$= \sum_{i<j}\left\{\left(a_j b_i - a_i b_j\right)^{2r}\left(a_j b_i + a_i b_j\right)\right\}$$

3. Application to polynomials.

Let $\mathrm{P}(x) = \prod_{i=1}^{n}(x-\mu_i)$ and $T_n = \sum_{i=1}^{n}|\mu_1|^n$ ($|\mu_i|$ is the module of μ_i) : For $x_i = |\mu_i|^p$ & $y_i = |\mu_i|^q$ we have

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}T_{kp+(m+1-k)q}T_{kq+(m+1-k)p} \geq 0$$

This inequality is true for all *p* and *q* such that *kp+(m+1-k)q* & *kq+(m+1-k)p* are integers.

We have the following relations for m = 1; 2; 3; 4; 5;…

$$\mathrm{m}=1 \rightarrow \quad T_{2p}T_{2q} - T_{p+q}^2 \geq 0$$
$$\mathrm{m}=2 \rightarrow \quad T_{3p}T_{3q} - 2T_{2p+q}T_{p+2q} \geq 0$$
$$\mathrm{m}=3 \rightarrow \quad T_{4p}T_{4q} - 4T_{3p+q}T_{p+3q} + 2T_{2p+2q}^2 \geq 0$$
$$\mathrm{m}=4 \rightarrow \quad T_{5p}T_{5q} + 2T_{3p+2q}T_{2p+3q} - 3T_{4p+q}T_{p+4q} \geq 0$$
$$\mathrm{m}=5 \rightarrow \quad T_{6p}T_{6q} + 15T_{4p+2q}T_{2p+4q} - 6T_{5p+q}T_{p+5q} - 10T_{3p+3q}^2 \geq 0$$
$$\vdots$$

References:

[1] Hardy G.H., Littlewood J.E., Polya G., *Inequalities.* Cambridge Univ. Press,1952.

[2] Mitrinovitch P.S.,Vasic P.M., *Analytic Inequalities*. Springer Verlag, 1970.

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