TENSOR PRODUCTS OF MODULES

by
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The notion of a tensor product of topological groups and modules is important in theory of topological groups, algebraic number theory. The tensor product of compact zero-dimensional modules over a pseudocompact algebra was introduced in [B] and for the commutative case in [GD], [L]. The notion of a tensor product of abelian groups was introduced in [H]. The tensor product of modules over commutative topological rings was given in [AU]. We will construct in this note the tensor product of a right compact R-module A_R and a left compact R-module ${}_RB$ over a topological ring R with identity. Some properties of tensor products are given.

Notation ω stands for the set of all natural numbers. If A is a locally compact group, K a compact subset of it and $\varepsilon>0$, then $T(K,\varepsilon) := \{\alpha\in A^* : \alpha(K) \subseteq \varphi(O\varepsilon)\}$, where φ is the canonical homomorphism of $\mathbb{R}$ on $\mathbb{R}/\mathbb{Z} = \mathbb{T}$. If m, n are natural numbers then $[m, n]$ stands for the set of all natural numbers x such that $m \le x \le n$.

Let R be a topological ring with identity and A_R, ${}_RB$ compact unitary right and left R-modules, respectively. A continuous function $\beta : A\times B \to C$, where C is a compact abelian group, is said to be R-balanced if it is linear on each variable, i.e., $\beta(a_1+a_2,b) = \beta(a_1,b) + \beta(a_2,b)$ and $\beta(a,b_1+b_2) = \beta(a,b_1) + \beta(a,b_2)$ for each a, a_1, $a_2\in A$, b, b_1, $b_2\in B$, and $\beta(ar, b) = \beta(a, rb)$ for each $a\in A$, $b\in B$, $r\in R$.

A pair (C, π) where C is a compact abelian group and $\pi : A\times B \to C$ is a R-balanced map is called a tensor product of A_R and ${}_RB$ provided for each compact abelian group D and each R-balanced mapping $\alpha : A\times B \to D$ there exists a unique continuous homomorphism $\alpha : C \to D$ such that the following diagram commutes,

$$\begin{array}{ccc} A\times B & \xrightarrow{\ \pi\ } & C \\ \alpha\searrow & & \swarrow\hat{\alpha} \\ & D & \end{array}$$

i.e., $\alpha = \hat{\alpha}\circ\pi$.

Remark. If A_R, ${}_RB$ are compact right and left R-modules, respectively, C a compact abelian group, $\pi : A\times B \to C$ a R-balanced mapping and $\alpha : C \to C_1$ a continuous homomorphism, then $\alpha\circ\pi : A\times B \to C_1$ is a R-balanced mapping.

Proposition. The tensor product, if it exists, is unique up to a topological isomorphism.

Proof. Let (C, π) be a tensor product of A_R and ${}_RB$. Then the subgroup C_1 of C generated by elements $\pi(a, b)$, $a\in A$, $b\in B$ is dense in C. Denote by p the canonical homomorphism of C on $C/\hat{C}_1$. Consider the trivial R-balanced mapping π_1 of $A\times B$ in $C/\hat{C}_1$, i.e., $\pi_1(a,b) = 0$ for every $a\in$A, $b\in$B. Then $\pi_1 = 0\circ\pi = p\circ\pi$. By the definition of a tensor product $p = 0$, hence $C = \hat{C}_1$.

Let now (C, π) and (C_1, π_1) be two tensor products of A_R and $_RB$. Then there exist continuous homomorphisms $\alpha : C \to C_1$ and $\beta : C_1 \to C$ such that $\pi_1 = \alpha\circ\pi$, $\pi = \beta\circ\pi_1$. Then $(\alpha\circ\beta)(\pi_1(a, b)) = \alpha(\beta(\pi_1(a, b))) = \alpha(\pi(a, b)) = \pi_1(a, b)$ for each $a\in A$, $b\in B$, hence $\alpha\circ\beta = 1_{C_1}$. In an analogous way, $(\beta\circ\alpha)(\pi(a, b)) = \beta(\alpha(\pi(a, b))) = \beta(\pi_1(a, b)) = \pi(a, b)$ for each $a\in A$, $b\in B$, hence $\beta\circ\alpha=1_C$. Therefore, C and C_1 are topologically isomorphic.

We will prove the existence of the tensor product for any compact right R-module A_R and any compact left R-module $_RB$. It will be denoted by $A\otimes_R B$.

Theorem 1. If A_R is a compact unitary right R-module and $_RB$ is a compact left unitary R-module over a topological ring R with identity then there exists the tensor product $A\otimes_R B$.

Proof. Let F be the discrete group of all R-balanced mappings f of $A\times B$ in T having the following properties:

i) $f(ar, b) = f(a, rb)$ for all $r\in R$, $a\in A$, $b\in B$

ii) there exists a neighborhood V of zero of R such that $f(av, b) = 0$ for all $v\in V$, $a\in A$, $b\in B$.

Consider the dual group $C = F^*$.

Define $\pi : A\times B \to C$ as follows: If $(a, b)\in A\times B$, then put $\pi(a, b)(f) := f(a, b)$ for each $f\in F$. It is easy to prove that π is a R-balanced mapping. Let, for example, $a\in A$, $r\in$R, $b\in B$. Then for each $f\in F$, $\pi(ar, b)(f) = f(ar, b) = $ f$(a, rb) = \pi(a, rb)(f)$, hence $\pi(ar, b) = \pi(a, rb)$.

We affirm that π is continuous. Let W be any neighborhood of zero of C. Then there is an $\varepsilon > 0$ and a finite subset K of F such that $T(K, \varepsilon)\subseteq W$. Since all $f\in K$ are continuous at (0, 0), there exist neighborhoods U, V of zeros of A and B, respectively, such that $f(U\times V)\subseteq \varphi(O_\varepsilon)$ for all $f\in K$. Then $\pi(U\times V)\subseteq W$. Indeed, if $f\in K$, $u\in U$, $v\in V$, then $\pi(u, v)(f) = f(u, v)\in \varphi(O_\varepsilon)$, and so $\pi(u, v)\in T(K, \varepsilon)$. We proved that $\pi(U\times V)\subseteq W$, hence π is continuous at (0,0),

Let $a\in A$, K a finite subset of F and $\varepsilon>0$. Since every $f\in K$ is continuous there exists a neighborhood V of zero of B such that $f(a, V)\subseteq \varphi(O_\varepsilon)$ for each $f\in K$. Then $\pi(a, V)\subseteq T(K, \varepsilon)$. Indeed, if $v\in V$, then for each $f\in K$, $\pi(a, v)(f) = f(a, v)\in \varphi(O_\varepsilon)$. Therefore $\pi(a, V)\subseteq T(K, \varepsilon)$. i.e., π is continuous at $(a, 0)$. By symmetry π is continuous at $(0, b)$, $b\in B$. We proved that π is a continuous R-balanced map.

We will prove now that C is the tensor product of A and B. Let $\alpha : A\times B \to X$ be a R-balanced map in a compact abelian group X. We define a homomorphism $\lambda : X^*\to F$ as follows: for every $\gamma\in X^*$, $\gamma\circ\alpha : A\times B \to \mathsf{T}$ is a R-balanced mapping of $A\times B$

in T, i.e., $\gamma\circ\alpha\in F$. Put $\lambda(\gamma) = \gamma\circ\alpha$, $\gamma\in X^*$. We claim that λ is a homomorphism. Indeed, let γ_1, $\gamma_2\in X^*$. Then for each $a\in A$, $b\in B$, $\lambda(\gamma_1+\gamma_2)(a, b) = (\gamma_1+\gamma_2)(\alpha(a, b)) = \gamma_1(\alpha(a,b)) + \gamma_2(\alpha(a, b)) = \lambda(\gamma_1)(a, b) + \lambda(\gamma_2)(a, b) = (\lambda(\gamma_1) + \lambda(\gamma_2))(a, b) \Rightarrow \lambda(\gamma_1+\gamma_2) = \lambda(\gamma_1) + \lambda(\gamma_2)$.

Let $\lambda^* : F^* \to X^{**}$ be the conjugate homomorphism for λ. Put $\hat{\alpha} : F^* \to X$, $\hat{\alpha} = \omega^{-1}\circ\lambda^*$, where ω is the canonical topological isomorphism of X on X^{**}. We affirm that $\alpha = \hat{\alpha}\circ\pi$. Indeed, fix $(a, b)\in A\times B$. Then $\alpha(a, b) = \hat{\alpha}(\pi(a, b)) \Leftrightarrow \alpha(a, b) = \omega^{-1}(\lambda^*(\pi(a, b))) \Leftrightarrow \omega(\alpha(a, b)) = \lambda^*(\pi(a, b)) \Leftrightarrow \omega(\alpha(a, b)) = \pi(a, b)\circ\lambda$. The last equality is true $\Leftrightarrow$ for each $\gamma\in X^*$, $\omega(\alpha(a, b))(\gamma) = (\pi(a, b)\circ\lambda)(\gamma) \Leftrightarrow \gamma(\alpha(a, b)) = \pi(a, b)(\lambda(\gamma)) \Leftrightarrow \gamma(\alpha(a, b)) = \pi(a, b)(\gamma\circ\alpha) \Leftrightarrow \gamma(\alpha(a, b)) = (\gamma\circ\alpha)(a, b) \Leftrightarrow \gamma(\alpha(a, b)) = \gamma(\alpha(a, b))$ which is true.

The uniqueness of $\hat{\alpha}$. It is sufficient to prove that the set $\{ \pi(a, b) : a\in A, b\in B\}$ generates $A\otimes_R B$ as a topological group. It is well known from the duality theory that if X is a locally compact abelian group and S a subgroup of X^* which separates points then S is dense in X^*. We affirm that the subgroup $D = \langle\{ \pi(a, b) : a\in A, b\in B \}\rangle$ separates points of F. Indeed, let $0 \neq \xi\in C$, then there exists $(a, b)\in A\times B$ such that $0 \neq \xi(a, b) = \pi(a, b)(\xi)$, i.e., D separates points of F. Therefore, α is unique.

We will denote below $\pi(a, b)$, where $a\in A$, $b\in B$ by $a\otimes b$.

Theorem 2. If A, B are zero-dimensional compact right and left R-modules then $A\otimes_R B$ is zero-dimensional.

Proof. Let $f\in F$; then f is a continuous R-balanced map of $A\times B$ in T. Let V be a neighborhood of zero of T which does not contain a non-zero subgroup. For every $a\in A$ there exist a neighborhood U_a and an open subgroup $V^{(a)}$ of B such that $f(U_a\times V^{(a)})\subseteq V$. There exist $a_1, \dots, a_n\in A$ such that $A = U_{a_1} \cup \dots \cup U_{a_n}$. Denote $V_0 = V^{a_1} \cap \dots \cap V^{a_n}$. We obtain immediately that $f(A, V_0)\subseteq V$, hence $f(A, V_0) = 0$.

Let $B = V_0 \cup (b_1+V_0) \cup \dots \cup (b_k+V_0)$. Then $f(A\times B)\subseteq f(A, b_1) + \dots + f(A, b_k)$. Each subset $f(A, b_1), \dots, f(A, b_k)$ is a compact zero-dimensional subgroup of T. Therefore $f(A, b_1) + \dots + f(A, b_k)$ is a finite subgroup of T. It follows that there exists $m\in\omega$ such that $mf = 0$, i.e., F is a torsion group. It is well known that F^* is zero-dimensional.

The author learned recently that a particular analogue of Theorem 2 was proved by Hofmann (see, [HM]).

Theorem 3. If A_R or $_RB$ is connected then $A\otimes_R B = 0$.

Proof. Assume that B is connected. Let V be a neighborhood of zero of T which does not contain non-zero subgroups. Fix $\xi\in C$. There exists a neighborhood V_0 of 0 of B

such that $\xi(A\times V_0) = 0$ (as in the proof of the previous theorem). Since B is generated by V_0, $\xi(A\times B)=0$. We obtained that $F = 0$, hence $C = 0$.

Let A_R be a compact right R-module and $_RB$ a compact left R-module over a topological ring R. If $X\subseteq A$, $Y\subseteq B$, then we will denote by $[X\otimes Y]$ the closure of the subgroup of $A\otimes_R B$ generated by elements of the form $\sum_{i=0}^{n} x_i \otimes y_i$, $x_i\in X$, $y_i\in Y$, $n\in\omega$.

Theorem 4. If A_R and $_RB$ are compact zero-dimensional left and right R-modules then the family $[U\otimes B]+[A\otimes V]$, where U runs all open subgroups of A and V runs all open subgroups of B is a fundamental system of neighborhoods of zero of $A\otimes_R B$.

Proof. The subgroup $\langle x\otimes y : x\in A, y\in B\rangle$ is dense in $A\otimes_R B$. Let U be an open subgroup of A and V an open subgroup of B. There exist finite symmetric subsets $F\subseteq A$, $K\subseteq B$ such that $A = F + U$, $B = K + V$. For each $x\in F$, $y\in K$, $u\in U$, $v\in V$, $(x + u)\otimes(y + v) = x\otimes y + x\otimes v + u \otimes y + u\otimes v$. Since A/U is finite, there exists $k\in\omega$ such that $kA\subseteq U$. Consider the finite subsets $H = \{ (lx)\otimes y : l\in[0, k\text{-}1], x\in F, y\in K\}$, $H_1= [l]H$. It is evidently that $C = H_1+ [U\otimes B] + [A\otimes B]$, hence $[U\otimes B] + [A\otimes B]$ is open.

Let W be an open subgroup of $A\otimes_R B$. By continuity of the mapping $\pi : A\times B \to A\otimes_R B$ and compactness of A and B there exist an open subgroup U of A and an open subgroup V of B such that $U\otimes B\subseteq W$, $A\otimes V\subseteq W\Rightarrow [U\otimes B] + [A\otimes V]\subseteq W$.

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