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ON SOME PROPERTIES OF THE SYMPLECTIC AND HAMILTONIAN MATRICES

by
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Abstract: In the first part of the paper the symplectic and Hamiltonian matrices are defined and some properties are pointed, and in the second part a relation between those two sets of matrices is proved.

1. Proprieties of symplectic and Hamiltonian matrices

For the beginning it will be introduced the square matrix $J \in \mathrm{M}_{2n\times 2n}(\,|\,)$ defined by

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \tag{1}$$

where: $0_n \in \mathrm{M}_{n\times n}(\,|\,)$ – zero matrix

$I_n \in \mathrm{M}_{n\times n}(\,|\,)$ – identity matrix

Remark 1

It is not very complicated to prove that the next properties of the matrix J are real (properties that will be very useful to prove some propositions that follow):

i) $J^t = -J$

ii) $J^1 = J^t$

iii) $J^t J = I_{2n}$

iv) $J^t J^t = - I_{2n}$

v) $J^2 = - I_{2n}$

vi) $\det J = \pm 1$

Now the definition of the symplectic and Hamiltonian matrices are given:

Definition 1

A matrix $A \in \mathrm{M}_{2n\times 2n}(\,|\,)$ is called symplectic if:

$$A^t J A = J \tag{2}$$

where $J \in M_{2n\times 2n}(\mathbb{R})$ is from (1).

We will denote by $SP(n, \mathbb{R}) \overset{not}{=} \{ A \in M_{2n\times 2n}(\mathbb{R}) \mid A^t J A = J \}$ the set of $2n \times 2n$ real symplectic matrices.

Definition 2

A matrix $A \in M_{2n\times 2n}(\mathbb{R})$ is called Hamiltonian if:

$$A^t J + JA = 0 \tag{3}$$

where $J \in M_{2n\times 2n}(\mathbb{R})$ is from (1).

We will denote by $sp(n, \mathbb{R}) \overset{not}{=} \{ A \in M_{2n\times 2n}(\mathbb{R}) \mid A^t J + JA = 0 \}$ the set of $2n \times 2n$ real Hamiltonian matrices.

In the next part some properties of those sets of matrices will be proved, for example:

Proposition 1

Let $A, B \in SP(n, \mathbb{R})$. The next relations are true:

a) A is nonsingular;

b) $A^{-1} = - J A^t J$;

c) $A^t, A^{-1}, AB \in SP(n, \mathbb{R})$.

Proof

a) From the definition of symplectic matrices we have $A^t J A = J \Rightarrow \det(A^t JA) = \det J \Rightarrow \det A^t \cdot \det J \cdot \det A = \det J \Rightarrow \det A^t \det A = 1 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1 \neq 0 \Rightarrow A$ is nonsingular.

b) $A \in SP(n, \mathbb{R}) \Rightarrow A^t J A = J \,\Big|\cdot A^{-1} \Rightarrow A^t J = J A^{-1} \Rightarrow J^{1} A^t J = A^{-1} \overset{J^{-1}=J^t}{\Rightarrow}$

$J^t A^t J = A^{-1} \overset{-J=J^t}{\Rightarrow} \Rightarrow A^{-1} = - J A^t J$. More, we have: $A = - J A^{-1} J$.

c) $A \in SP(n, \mathbb{R}) \Rightarrow A^t J A = J \,\big|\cdot J \Rightarrow A^t J A J = J^2 = - I_{2n} \,\Big|\cdot A^t \Rightarrow A^t J A J A^t = - A^t$ Now, multiplying at the left side by $(A^t)^{-1}$, we will obtain: $J A J A^t = - I_{2n}$. Multiplying again at the right side by J^{-1}: $A J A^t = - J^{-1} = -(-J) = J \Rightarrow A J A^t = J \Rightarrow (A^t)^t J A^t = J \Rightarrow A^t \in SP(n, \mathbb{R})$

We will proof now that $A^{-1} \in SP(n, \mathbb{R})$. Using the relation $A^{-1} = - J A^t J$ (from the point b)) we have successive:

$(A^{-1})^t J A^{-1} = (- J A^t J)^t J (- J A^t J) =$

$= J^t \ (JA)^t JJA^tJ =$ $\qquad$ $J^2 = -I_{2n}$

$= J^t \ A \ J^t \ (-I_{2n}) A^t J =$

$= -J^t \ A \ J^t \ A^t J =$ $\qquad$ $A \ J^t \ A^t = (A^t J A)^t = J^t$

$= -J^t \ J^t \ J =$ $\qquad$ $J^t \ J^t = -I_{2n}$

$= J \quad \Rightarrow \quad A^{-1} \in$ SP(n, |)

We will proof now that $AB \in$ SP(n, |):

$(AB)^t J AB = B^t A^t J A B =$ $\qquad$ $A^t J A = J$

$= B^t J B =$

$= J \quad \Rightarrow \quad AB \in$ SP(n, |)

Consequence. The set of symplectic matrices SP(n, |) is a subgroup of the set of nonsingular matrices GL(n, |) reported to multiplication. Indeed we have $AB \in$ SP(n, |), $\forall$ $A, B \in$ SP(n, |), and $AB^{-1} \in$ SP(n, |), $\forall$ A, B $\in$ SP(n, |).

Proposition 2

Let $A \in$ SP(n, |) and $p_A(x)$ - the characteristic polynomial of the matrix A. If $p_A(c) = 0$, then $p_A(\frac{1}{c}) = p_A(\bar{c}) = p_A(\frac{1}{\bar{c}}) = 0$, where $c \in$ |.

Proof

$p_A(x) = \det(A - x I_{2n}) =$ $\qquad$ $A = -J A^{-1} J$

$= \det(-J A^{-1} J - x I_{2n}) =$ $\qquad$ $I_{2n} = -J^2$

$= \det(J(-A^{-1}) J + x J^2) =$

$= \det(J(-A^{-1} + x I_{2n}) J) =$

$= \det J \det(-A^{-1} + x I_{2n}) \det J =$ $\qquad$ $\det J = \pm 1$

$= \det(-A^{-1} + x I_{2n}) =$

$= \det(-A^{-1} + xAA^{-1}) =$

$= \det(-I_{2n} + xA) \det(A^{-1}) =$

$= \pm \det(xA - I_{2n}) = \pm x^{2n} \det(A - \frac{1}{x} I_{2n}) = \pm x^{2n} p_A(\frac{1}{x}) \Rightarrow$

$\Rightarrow x^{2n} p_A(\frac{1}{x}) = 0 \Rightarrow p_A(\frac{1}{x}) = 0 \Rightarrow p_A(\frac{1}{c}) = 0 \Rightarrow p_A(\bar{c}) = 0 \Rightarrow p_A(\frac{1}{\bar{c}}) = 0.$

Proposition 3

The next relations are equivalent:

a) A is a Hamiltonian matrix;

b) $A = JS$, where $S = S^t$;

c) $(JA)^t = JA$.

Proof

„ a $\Leftrightarrow$ b "

$A = J\,J^{-1}A \Leftrightarrow \underline{A = J(-J)A} \overset{A\in sp(n,R)}{\Leftrightarrow} (J(-JA))^t J + JA = 0 \Leftrightarrow (-JA)^t J^t J = -JA \overset{J^tJ=I_{2n}}{\Leftrightarrow} (-JA)^t = -JA \Leftrightarrow \underline{J(-JA)^t = A}$

If $-JA = S \Leftrightarrow A = J(-JA) = J(-JA)^t \Leftrightarrow A = JS = JS^t \Rightarrow S = S^t$.

„ a $\Leftrightarrow$ c "

$A^t J + JA = 0 \Leftrightarrow A^t J = -JA \overset{()^t}{\Leftrightarrow} (A^t J)^t = (-JA)^t \Leftrightarrow J^t A = -(JA)^t \Leftrightarrow -JA = -(JA)^t \Leftrightarrow (JA)^t = JA$

Proposition 4

Let $A, B \in$ sp(n, $\mathbb{R}$). The next relations are true:

a) $A + B \in$ sp(n, $\mathbb{R}$);

b) $\alpha A \in$ sp(n, $\mathbb{R}$), $\alpha \in \mathbb{R}$;

c) $[A, B] \in$ sp(n, $\mathbb{R}$), where $[A, B] \overset{def}{=} AB - BA$

Proof

a) Because A and B are Hamiltonian matrices it results that $A^t J + JA = 0$ respectively $B^t J + JB = 0$. By adding those two relations we will obtain:

$(A^t + B^t) J + J(A + B) = 0 \Leftrightarrow (A + B)^t J + J(A + B) = 0 \Leftrightarrow A + B \in$ sp(n, $\mathbb{R}$).

ii) $A^t J + JA = 0 \mid \alpha \Leftrightarrow A^t J\alpha + JA\alpha = 0 \Leftrightarrow A^t\alpha J + J(A\alpha) = 0 \Leftrightarrow$
$\Leftrightarrow (A\alpha)^t J + J(A\alpha) = 0 \Leftrightarrow \alpha A \in$ sp(n, $\mathbb{R}$).

iii) We will prove that $[A, B] = JM$, where $M = M^t$.

We know that $A = JS$ and $B = JR$, where $S = S^t$ and $R = R^t$.

$[A, B] = AB - BA = JSJR - JRJS = J(SJR - RJS)$, from where, making the notation $SJR - RJS = M$ we will obtain $[A, B] = JM$.

Now we will show that $M = M^t$

$M^t = (SJR - RJS)^t = (SJR)^t - (RJS)^t = R^tJ^tS^t - S^tJ^tR^t = -RJS + SJR = SJR - RJS = M$

Consequence. (sp(n, $\mathbb{R}$), $[\bullet, \bullet]$) is a Lie algebra.

Proof. We will prove the necessary properties of the bracket $[\bullet, \bullet]$: bilinearity, antisymmetry and Jacobi's relation.

i) $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$ - evidently $\Rightarrow$ the operation is bilinear.

ii) $[A, B] = AB - BA = -(BA - AB) = -[B, A]$ $\Rightarrow$ the operation is antisymmetric.

iii) Jacobi's relation is satisfied:

$[[A, B], C] + [[C, A], B] + [[B, C], A] = [AB - BA, C] + [CA - AC, B] + [BC - CB, A] = ABC - BAC - (CAB - CBA) + CAB - ACB - (BCA - BAC) + BCA - CBA - (ABC - ACB) = 0$

Proposition 5

Let A $\in$ sp(n, $\mathbb{R}$) and $p_A(x)$ - the characteristic polynomial of the matrix A. Then:

a) $p_A(x) = p_A(-x)$

b) if $p_A(c) = 0$, then $p_A(-c) = p_A(\bar{c}) = p_A(-\bar{c}) = 0$, where $c \in \mathbb{C}$.

Proof

a) $p_A(x) = \det(A - x I_{2n})$, but $A = J A^t J$ $\Rightarrow$

$$\begin{aligned} p_A(x) &= \det(J A^t J - x I_{2n}) && A = -J A^{-1} J \\ &= \det(J A^t J - J x J) = \\ &= \det(J(A^t + x I_{2n})J) = \\ &= \det J \det(A^t + x I_{2n}) \det J = && \det J = \pm 1 \\ &= \det(A^t + x I_{2n}) = \det(A^t + x I_{2n}{}^t) = \\ &= \det(A + x I_{2n})^t = \\ &= \det(A + x I_{2n}) = \det(A - (-x) I_{2n}) = p_A(-x) \end{aligned}$$

b) $p_A(c) = 0 \overset{a)}{\Rightarrow} p_A(-c) = 0$.

$p_A(x)$ is a real coefficients polynomial $\Rightarrow p_A(\bar{c}) = 0 \overset{a)}{\Rightarrow} p_A(-\bar{c}) = 0$.

2.A relation between the sets sp(n, $\mathbb{R}$) and SP(n, $\mathbb{R}$)

Theorem:

Let $A \in M_{2n\times 2n}(\mathbb{R})$. The next relations are equivalent:

a. $A \in$ sp (n, $\mathbb{R}$)
b. exp (At) $\in$ SP (n, $\mathbb{R}$)

Proof

„a $\Rightarrow$ b"

$A \in$ sp (n, $\mathbb{R}$) $\Leftrightarrow A^t J + JA = O$

Let $U(t) = (\exp(At))^{t} J \exp(At)$

But, on the other hand we have:

$$\frac{dU}{dt} = [\frac{d}{dt}\exp(At)]^{t} J \exp(At) + [\exp(At)]^{t} J [\frac{d}{dt} \exp(At)]$$
$$= [A \exp(At)]^{t} J \exp(At) + [\exp(At)]^{t} J A (\exp(At))$$
$$= [\exp(At)]^{t} A^{t} J \exp(At) + [\exp(At)]^{t} JA \exp(At)$$
$$= [\exp(At)]^{t} [A^{t} J + J A] \exp(At) = 0.$$

$\Rightarrow U(t)$ – constant. (*)

But $U(0) = (\exp(A \text{ A } O))^{t} J \exp(A \text{ A } O) = J$ (**)

From (*) and (**) $\Rightarrow U(t) = 0 \Rightarrow (\exp(At))^{t} J \exp(At) = J \Rightarrow \exp(At) \in$ SP (n, |).

„b $\Rightarrow$ a"

$\exp(At) \in$ sp (n, |) $\Rightarrow (\exp(At))^{t}$ A J A $\exp(At)$ = J. $\Rightarrow$

$$\Rightarrow \frac{d}{dt}[(\exp(At))^{t} J \text{ A } (\exp(At))] = 0 \Rightarrow$$

$$\Rightarrow [\frac{d}{dt}\exp(At)]^{t} J \exp(At) + (\exp(At))^{t} J [\frac{d}{dt}\exp(At)] = 0$$

$$\Rightarrow (A \exp(At))^{t} J \exp(At) + (\exp(At))^{t} J A \exp(At) = 0$$

$$\Rightarrow (\exp(At))^{t} A^{t} J \exp(At) + (\exp(At))^{t} J A \exp(At) = 0$$

$$\Rightarrow (\exp(At))^{t} [A^{t} J + JA] \exp(At) = 0$$

Multiplying at the right by $((\exp(At))^{t})^{-1}$ and at the left by $(\exp(At))^{-1}$ we will obtain:

$A^{t} J + JA = 0 \Rightarrow A \in$ sp (n, |).

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