INTEGRAL OPERATORS ON THE TUCD(α)-CLASS

by
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Abstract. In this paper we present a few univalency conditions for various integral operators on class $TUCD(\alpha)$. At first, we make a brief presentation of class $TUCD(\alpha)$ and of some of its properties, as well as a number of matters connected to some integral operators studied on this class.

Introduction.

Let $U = \{z : |z < 1|\}$ be the unit disk, respectively the class of olomorphic functions on the unit disk, denoted by $H(U)$, $A = \{f \in H(U) : f(z) = z + a_2 z^2 + ...\}$ the class of the analytical functions in the unit disk and the set $D = \{\phi : \phi \text{ is analytic in } U, \phi(z) \neq 0, \forall z \in U, \phi(0) = 1\}$.

We consider the class of starlike functions of the order α on the unit disk, the univalent functions with the property $\text{Re}\dfrac{zf'(z)}{f(z)} > \alpha$, respectively, the class of the convex functions on the order α in unit disk, the univalent functions with the property $\text{Re}\dfrac{zf''(z)}{f'(z)} + 1 > \alpha$. We denote these with $S^*(\alpha)$ and $K(\alpha)$, and for $\alpha = 0$ we obtain the class of the starlike funcions S^*, respectively the class of convex functions K.

We say that the function f with the form $f(z) = z + a_2 z^2 + ...$ belongs to the class $UCD(\alpha)$, $\alpha \geq 0$, if

$$\text{Re}\, f'(z) \geq \alpha |zf''(z)|,\ z \in U. \qquad (1)$$

If $f \in UCD(\alpha)$, then the following relation is true

$$|f'(z)| \geq \alpha |zf''(z)|, z \in U \qquad (2)$$

which can also be written as

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1}{\alpha}. \qquad (3)$$

In [3] we studied the class of univalent functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \ge 0. \quad (4)$$

We denote $TUCD(\alpha)$, the functions from $UCD(\alpha)$ of the form (4). Next we consider all the functions from this paper of the form (4).

Theorem 1.1. [3] A function of the form (1) belongs to the class $TUCD(\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k[1+\alpha(k-1)] \cdot a_k \le 1.$$

Remark 1.2. [3]

a) A function of the form (4) is starlike if and only if $f \in TUCD(0)$.
b) A function of the form (4) is convex if and only if $f \in TUCD(1)$.
c) A function of the form (4) is convex by the order 1/2 if and only if $f \in TUCD(2)$.
d) A function of the form (4) is uniform convex if and only if $f \in TUCD(2)$.
e) A function $f \in TUCD(\alpha)$ if and only if f can be written as

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \qquad (5)$$

where $\lambda_k \ge 0, \sum_{k=1}^{\infty} \lambda_k = 1$, $f_1(z) = z$ and $f_k(z) = z - \frac{z^k}{k[1+\alpha(k-1)]}, k = 2,3,...$

Theorem 1.3. [2] Let $\alpha, \beta, \gamma, \delta$ be real constants that satisfy the conditions $\alpha \ge 0, \beta > 0$ $(\alpha+\delta) = (\beta+\gamma) > 0$. Let ρ_0 be so that $-\frac{\operatorname{Re}\gamma}{\operatorname{Re}\beta} \equiv \rho_0 < \rho < 1$ and we suppose that $\rho \in [\rho_0, 1]$ exists, so that $0 \le w(\rho)$, where $w(\rho) = \frac{1}{\operatorname{Re}\beta} \inf\{H(z) : |z| < 1\}$ and $H(z) = \frac{(1-z)^{2(\rho-1)\operatorname{Re}\beta}}{\int_0^1 t^{\beta+\gamma-1}(1+tz)^{2(\rho-1)\operatorname{Re}\beta} dt}$.

Let φ and $\phi \in D$ satisfy the conditions:

$$\delta + \operatorname{Re}\frac{z\varphi'(z)}{\varphi(z)} \geq \beta\rho + \gamma \qquad (6)$$

and

$$\operatorname{Re}\frac{z\phi'(z)}{\phi(z)} \leq \beta w(\rho) \qquad (7)$$

If $I(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}$, then $I(S^*) \subset S^*$.

Theorem 1.4. [1] If $0 \leq \alpha \leq 1, \alpha \leq \beta$ and if $f \in S^*$, then the function

$$F(z) = \left[z^{\beta-1}\int_0^z \left(\frac{f(z)}{t}\right)^{\alpha} dt\right]^{\frac{1}{\beta}} = z + \dots \qquad (8)$$

is also an element of S^*.

Theorem 1.5. [1] If $\alpha > 0, \eta \geq 0, \gamma + \eta \geq 0$ and if $f \in S^*$, then the function

$$F(z) = \left[\frac{\alpha+\gamma+\eta}{z^{\gamma}}\int_0^z f^{\alpha}(t)t^{\gamma+\eta-1}dt\right]^{\frac{1}{\alpha+\eta}} = z + \dots \qquad (9)$$

is starlike in U.

Theorem 1.6. [2] Let $\alpha, \beta, \gamma, \delta$ and σ real number which satisfy $0 \leq \alpha, 0 < \beta, 0 \leq \sigma$ and $\beta + \gamma = \alpha + \delta > 0$. Let ρ_0 defined in the theorem 1.3. and we suppose that exists $\rho \in [\rho_0, 1]$ so that $\delta - \frac{\sigma}{2} \geq \beta + \gamma$ and $w(\rho) > 0$ where $w(\rho)\operatorname{Re}\beta \equiv Inf\{\operatorname{Re} H(z) : |z| < 1\}$. Let $\phi \in D$ which satisfies

$\operatorname{Re}\frac{z\phi'(z)}{\phi(z)} \le \beta w(\rho)$. If $f \in S^*, g \in K$ then $I(f,g) \in S^*$, where $I(f,g)$ is defined

$$I(f,g)(z) = \left[\frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t) g^\sigma(t) t^{\delta-\sigma-1} dt\right]^{\frac{1}{\beta}}. \qquad (10)$$

Corollary 1.7. [1] Let $0 \le \alpha, 0 < \beta, 0 \le \sigma$, $\gamma > -\beta, \beta + \gamma = \alpha + \delta$, and $\delta + \frac{\alpha}{2} - \frac{\sigma}{2} \ge \gamma$. Let $\phi \in D$ which satisfies $\operatorname{Re}\frac{z\phi'(z)}{\phi(z)} \le \beta w(0)$. If $f, g \in K$ then $I(f,g) \in S^*$, where $I(f,g)$ is defined in (10).

Main results

Theorem 2.1. Let ξ and δ complex number, $\xi + \delta > 0, \phi \in D$.
If $f \in A$ is of the form (4) and

$$\frac{\xi z f'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec Q_{\xi+\delta}, \qquad (11)$$

while

$$F_1(f) = \left[(\xi+\delta)\int_0^z f^\xi(t) \cdot t^{\delta-1} \cdot \phi(t) dt\right]^{\frac{1}{\xi+\delta}}. \qquad (12)$$

Then $F_1(f) \in TUCD(0)$.
Theorem 2.2. Let

$$F_2(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \cdot \phi(z)}\int_0^z f^\xi(t) \cdot \phi(t) \cdot t^{\delta-1} dt\right]^{1/\beta}. (13)$$

If $\xi, \beta, \gamma, \delta \in R$, $\xi \ge 0$, $\beta > 0$, $\beta + \gamma = \xi + \delta$ şi ρ_0 verifies the relation

$$-\frac{\operatorname{Re}\gamma}{\operatorname{Re}\beta} \equiv \rho_0 \le \rho < 1 \qquad (14)$$

and we suppose that exists $\rho \in [\rho_0, 1]$ so that $0 \le w(\rho)$, where w verifies the relation

$$\operatorname{Re}\beta \frac{zG'(z)}{G(z)} \ge w(\rho) \cdot \operatorname{Re}\beta = Inf\{\operatorname{Re} H(z) \| z| < 1\}, \qquad (15)$$

where

$$G(z) = I_{\beta,\gamma}(g)(z) = \left[(\beta+\gamma)z^{-\gamma}\int_0^z g^{\beta}(t)\cdot t^{\gamma-1}dt\right]^{1/\beta},\ g \in A \qquad (16)$$

and

$$H(z) = \frac{(1-z)^{2(\rho-1)\operatorname{Re}\beta}}{\int_0^1 t^{\beta+\gamma-1}(1+tz)^{2(\rho-1)\operatorname{Re}\beta}dt} - \gamma, \qquad (17)$$

φ and $\phi \in D$ satisfy the relations

$$\delta + \operatorname{Re}\frac{z\varphi'(z)}{\varphi(z)} \geq \beta\rho + \gamma \qquad (18)$$

and

$$\operatorname{Re}\frac{z\phi'(z)}{\phi(z)} \leq \beta \cdot w(\rho) \qquad (19)$$

then $F_2(f) \in TUCD(0)$.

Theorem 2.3. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ **,** $a_k \geq 0$

a)If $f \in S^* \Rightarrow F_3(f) \in TUCD(1)$, where

$$F_3(f)(z) = \int_0^z \frac{f(t)}{t}dt \qquad (20)$$

is the Alexander operator.

Proof.

We know that if $f \in S^* \Rightarrow F_3(f) \in K$. Considering the remark 1.2. b), we obtain $F_3(f) \in K \Leftrightarrow F_3(f) \in TUCD(1)$.

b) $f \in S^* \Rightarrow F_4(f) \in TUCD(0)$, where

$$F_4(f)(z) = \frac{2}{z}\int_0^z f(t)dt \qquad (21)$$

is the Libera operator.

Proof.

We know that if $f \in S^* \Rightarrow F_4(f) \in S^*$. Considering the remark 1.2. a), we obtain $F_4(f) \in S^* \Leftrightarrow F_4(f) \in TUCD(0)$.

c) $f \in S^* \Rightarrow F_5(f) \in TUCD(0)$, where

$$F_5(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) \cdot t^{\gamma-1} dt \qquad (22)$$

$\gamma \geq -1$ is the Bernardi operator.

Proof.
We know that if $f \in S^* \Rightarrow F_5(f) \in S^*$. Considering the remark 1.2. a), we obtain $F_5(f) \in S^* \Leftrightarrow F_5(f) \in TUCD(0)$.

Theorem 2.4. Let

$$F_6(f)(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\beta dt \qquad (23)$$

and $0 \leq \beta \leq 1$.

If $f \in S^*$ then $F_6(f) \in TUCD(0)$.

Proof.
Considering the theorem 1.4, for $\beta = 1, \alpha \equiv \beta$ we obtain $f \in S^* \Rightarrow F_6(f) \in S^*$. According to the remark 1.2. a) we obtain $F_6(f) \in TUCD(0)$.

Theorem 2.5. Let

$$F_7(f)(z) = \left[\beta \int_0^z f^\beta(t) \cdot t^{-1} dt\right]^{1/\beta}, \ \beta > 0. \qquad (24)$$

If $f \in S^*$ then $F_7(f) \in TUCD(0)$.

Proof.
Applying the theorem 1.5 for this integral operator for $\gamma = 0, \eta = 0, \beta = \alpha$ we obtain $f \in S^* \Rightarrow F_7(f) \in S^*$. But the remark 1.2, a) we obtain $F_7(f) \in TUCD(0)$.

Theorem 2.6. Let

$$F_8(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z f^\beta(t) \cdot t^{-1} dt\right]^{1/\beta}, \ \gamma, \beta = 1,2,3\ldots \ (25)$$

If $f \in S^*$ then $F_8(f) \in TUCD(0)$.

Proof.
We have $f \in S^* \Rightarrow F_8(f) \in S^*$, according to the theorem 1.5. Applying the remark 1.2, a), we obtain $F_8(f) \in TUCD(0)$.
Theorem 2.7. Let

$$F_9(f,g)(z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z \left[\frac{f(t)}{t} \right]^\xi \left[\frac{g(t)}{t} \right]^\delta \cdot t^{\xi+\delta-1} dt \right]^{1/\beta} . \quad (26)$$

If $f \in S^*$, $g \in K$, then $F_9(f,g) \in TUCD(0)$.
Proof.
In [2] we proved that if $f \in S^*, g \in K \Rightarrow F_9(f,g) \in S^*$. Applying the remark 1.2 a) we obtain $F_9(f,g) \in TUCD(0)$.
Theorem 2.8. Let

$$F_{10}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \cdot \phi(z)} \int_0^z f^\xi(t) \cdot \varphi(t) \cdot t^{\delta-1} dt \right]^{1/\beta}, \ f \in A \quad (27)$$

and the following conditions are satisfied:
$\varphi, \phi \in D, \xi, \beta, \gamma, \delta$ the complex numbers with the properties,

$$\beta > 0, \xi+\delta = \beta+\gamma, \ \mathrm{Re}(\xi+\delta) > 0, \ \mathrm{Re}\left[\frac{z\phi'(z)}{\phi(z)} + \gamma \right] \le 0 . \quad (28)$$

Then $F_{10}(f) \in TUCD(0)$.
Proof.
If $f \in A$ satisfies the conditions of this theorem then $F_{10}(f) \in S^*$, the result proved in [2].
Applying the remark 1.2, a) we obtain $F_{10}(f) \in TUCD(0)$.

Theorem 2.9. Let ξ şi δ complex number, $\xi+\delta > 0, \phi \in D$.
If $f \in A$ is the form (4) and

$$\frac{\xi z f'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec Q_{\xi+\delta}, \quad (29)$$

and

$$F_{11}(f) = \left[(\xi+\delta) \int_0^z f^\xi(t) \cdot t^{\delta-1} \cdot \phi(t) dt \right]^{\frac{1}{\xi+\delta}} . \quad (30)$$

Then $F_{11}(f) \in TUCD(0)$.

Proof.
If $f \in A$ are the form (4) and verifies the conditions of the theorem 2.1, we obtain $F_{11}(f) \in S^* \Leftrightarrow F_{11}(f) \in TUCD(0)$, also considering the remark 1.2, a).

Theorem 2.10. Let

$$F_{12}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma} \cdot \phi(z)} \int_0^z f^{\xi}(t) \cdot \phi(t) \cdot t^{\delta-1} dt \right]^{1/\beta}. \qquad (31)$$

If $\xi, \beta, \gamma, \delta \in R$, $\xi \geq 0$, $\beta > 0$, $\beta+\gamma = \xi+\delta$ and ρ_0 verifies the relation (14) and suppose that exists $\rho \in [\rho_0, 1]$ so that $0 \leq w(\rho)$, where w verifies the relation

$$\operatorname{Re} \beta \frac{zG'(z)}{G(z)} \geq w(\rho) \cdot \operatorname{Re} \beta = Inf \{ \operatorname{Re} H(z) \| z | < 1 \}, \qquad (32)$$

where

$$G(z) = I_{\beta,\gamma}(g)(z) = \left[(\beta+\gamma) z^{-\gamma} \int_0^z g^{\beta}(t) \cdot t^{\gamma-1} dt \right]^{1/\beta}, \ g \in A \qquad (33)$$

and

$$H(z) = \frac{(1-z)^{2(\rho-1)\operatorname{Re}\beta}}{\int_0^1 t^{\beta+\gamma-1} (1+tz)^{2(\rho-1)\operatorname{Re}\beta} dt} - \gamma, \qquad (34)$$

φ şi $\phi \in D$ and satisfy the relations

$$\delta + \operatorname{Re} \frac{z\varphi'(z)}{\varphi(z)} \geq \beta\rho + \gamma \qquad (35)$$

and

$$\operatorname{Re} \frac{z\phi'(z)}{\phi(z)} \leq \beta \cdot w(\rho) \qquad (36)$$

then $F_{12}(f) \in TUCD(0)$.

Proof.
Let $f \in A$ and satisfies the above conditions according to theorem 1.3 and the remark 1.2, a) we obtain $F_{12}(f) \in S^* \Leftrightarrow F_{12}(f) \in TUCD(0)$.

Theorem 2.11. Let the integral operator $F_{12}(f)$ defined in the theorem 2.10, which verifies the conditions of the theorem 2.10, and the condition (35) from theorem 2.10 becomes the following:

$$\frac{\xi}{2}+\delta+\mathrm{Re}\frac{z\varphi'(z)}{\varphi(z)}\geq\beta\rho+\gamma\,, \qquad (37)$$

then if $f\in K\Rightarrow F_{12}(f)\in TUCD(0)$.

Proof.

If $f\in K$, has the form (4) and verifies the conditions of this theorem we obtain $F_{12}(f)\in S^*$. Applying the remark 1.2, a) we obtain $F_{12}(f)\in TUCD(0)$.

Theorem 2.12. Let

$$F_{13}(f,g)=\left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\xi}(t)\cdot g^{\sigma}(t)\cdot t^{\delta-\sigma-1}dt\right]^{1/\beta}, \qquad (38)$$

ξ,β,γ,δ and $\sigma\in R$, $\xi\geq 0$, $\beta>0$, $\sigma\geq 0$, $\beta+\gamma=\xi+\delta>0$. Let ρ_0 from the theorem 2.10 and suppose that $\rho\in[\rho_0,1]$ so that

$$\delta-\frac{\sigma}{2}\geq\beta+\gamma\,,\ w(\rho)>0\,, \qquad (39)$$

w given in the treorem 2.10.

Let $\varphi\in D$, so that

$$\mathrm{Re}\frac{z\varphi'(z)}{\varphi(z)}\leq\beta\cdot w(\rho). \qquad (40)$$

If $f\in S^*, g\in K$, then $F_{13}(f,g)\in TUCD(0)$.

Proof.

We consider $f\in S^*, g\in K$ of the form (4). According to the theorem 1.7, proved in [2] we obtain $F_{13}(f,g)\in S^*$. Applying the remark 1.2, a) we obtain $F_{13}(f,g)\in TUCD(0)$.

Theorem 2.13. Let

$$F_{14}(f,g)(z)=\left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\xi}(t)\cdot g^{\sigma}(t)\cdot t^{\delta-\sigma-1}dt\right]^{1/\beta}, \qquad (41)$$

ξ,β,γ,δ and $\sigma\in R$, $\alpha\geq 0$, $\beta>0$, $\sigma\geq 0$, $\beta+\gamma=\xi+\delta>0$, $\rho_0, w(\rho)$ as in the theorem 2.10 and exists $\rho\in[\rho_0,1]$ so that

$$\rho+\frac{\xi}{2}-\frac{\sigma}{2}\geq\beta\rho+\gamma,\ w(\rho)\geq 0. \qquad (42)$$

Let $\phi\in D$ so that

$$\operatorname{Re}\frac{z\phi'(z)}{\phi(z)}\leq\beta w(\rho). \qquad (43)$$

If $f,g\in K$ of the form (4) then $F_{14}(f,g)\in TUCD(0)$.

Proof.

If $f,g\in K$ according to the corrolary 1.6, the result proved in [1] we obatin $F_{14}(f,g)\in S^{*}$. Applying the remark 1.2, a) we obatin $F_{14}(f,g)\in TUCD(0)$.

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