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PERFECT TRANSVERSALS IN SOME GRAPH PRODUCTS

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Abstract. A perfect transversal of a graph is a set of vertices that meets any maximal clique, or any maximal stable set of the graph in exactly one vertex. In this paper we give some properties of perfect transversals in graph products. One of these properties is the characterization of the existence of perfect transversals in some graph products via perfect transversals of the factors. As an application we give a characterization of strongly perfectness of such graph products. We have also got similar results for the graph itself using graph decomposition. In order to obtain these results we have strongly used and we have given some new properties of graph homomorphisms and quasi-homomorphisms.

Keywords: perfect transversal, stable, clique, graph homomorphism, strongly perfectness.

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1. Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. $\overline{G}$ designates the complement of G. If $e = xy \in E$, we shall also write $x \sim y$, and $x \nsim y$ whenever x, y are not adjacent in G. If $A \subseteq V$, then $G[A]$ is the subgraph of G induced by $A \subseteq V$; shortly, we use *subgraph* for *induced subgraph*, and by $H \subseteq G$ we mean that H is a subgraph of G. If $W \subseteq E$, then by $G - W$ we denote the graph $(V, E - W)$. By (A, B) we mean the set $\{ab : a \in A, b \in B, ab \in E\}$, where $A, B \subset V, A \cap B = \emptyset$. We write $A \sim B$ whenever $ab \in E$ holds for all $a \in A$ and $b \in B$, and $A \nsim B$ whenever $(A, B) = \emptyset$.

By P_n, C_n and K_n we denote a chordless path on $n \geq 3$ vertices, the chordless cycle on $n \geq 3$ vertices, and the complete graph on $n \geq 1$ vertices, respectively.

An *independent* set in G is a set of mutually non-adjacent vertices, a *stable set* in G is a maximal (with respect to set inclusion) independent set, and the *stability number* $\alpha(G)$ of G is the maximum cardinality of a stable set. By $\mathcal{S}(G)$ we denote the family of all stable sets of G, and $\mathcal{S}_\alpha(G) = \{S : S \in \mathcal{S}(G), |S| = \alpha(G)\}$. A *complete set* in G is a subset A of $V(G)$ that induces a complete subgraph in G. A *clique* in G is a maximal complete set in G, and the *clique number* is $\omega(G) = \alpha(\overline{G})$. By $\mathcal{C}(G)$ we denote the family of all cliques of G, and $\mathcal{C}_\omega(G) = \{C : C \in \mathcal{C}(G), |C| = \omega(G)\}$. Clearly, $\mathcal{C}(G) = \mathcal{S}(\overline{G}), \mathcal{C}_\omega(G) = \mathcal{S}_\alpha(\overline{G}), \mathcal{S}_\alpha(G) \subseteq \mathcal{S}(G)$ and $\mathcal{C}_\omega(G) \subseteq \mathcal{C}(G)$ hold for any graph G.

The *chromatic number* and the *clique covering number* of G are, respectively, $\chi(G)$ and $\theta(G)$, where $\theta(G) = \chi(\overline{G})$.

A graph G is *perfect* if $\alpha(H) = \theta(H)$ (or, equivalently, by Perfect Graph Theorem, $\chi(H) = \omega(H)$) holds for every induced subgraph H of G.

Definition 1.*Let M be a non-empty set, and $\mathcal{F} = \{M_i : i \in I\}$ be a family of subsets of M. A subset T of M is called a transversal of $\mathcal{F}$ if $T \cap M_i \neq \emptyset$ for any $i \in I$. A transversal T of M is called perfect if $|T \cap M_i| = 1$ for any $i \in I$.*

A perfect transversal of $\mathcal{S}(G)$ or $\mathcal{C}(G)$ has a precise structure [6]:

For any graph G, a transversal T of $\mathcal{S}(G)$ is perfect if and only if $T \in \mathcal{C}(G)$.[1]

This motivates the names of *clique transversal* and *stable transversal* for a perfect transversal of $\mathcal{S}(G), \mathcal{C}(G)$, respectively, and the notations: $\mathcal{T}_c(G)$ for the family of all complete transversals of G, and $\mathcal{T}_s(G) = \mathcal{T}_c(\overline{G})$ for the family of all stable transversals of G. A perfect transversal of G is either a stable transversal or a clique transversal of G, according to the context.

If G and H are graphs, then by $f : G \to H$ we mean a function $f : V(G) \to V(H)$. As usual, if $X \subseteq V(G)$, $Y \subseteq V(H)$, then $f(X) = \{f(x) : x \in V(G)\}$ and $f^{-1}(Y) = \{x : x \in V(G),\ f(x) \in Y\}$; for short, $f^{-1}(\{y\})$, $y \in V(H)$ is denoted by $f^{-1}(y)$. If $\mathcal{W} \subseteq \mathcal{P}(V(G))$ and $\mathcal{Y} \subseteq \mathcal{P}(V(H))$ such that $f(X) \in \mathcal{Y}$, for all $X \in \mathcal{W}$, then f generates a function $F : \mathcal{W} \to \mathcal{Y}$, defined by $F(X) =$

[1]Let us notice that there are graphs G whose $\mathcal{S}_\alpha(G)$ have perfect transversals that do not induce complete subgraphs; e.g., if

$$V(C_{2n}) = \{v_i : 1 \leq i \leq 2n\}, E(C_{2n}) = \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{v_1 v_{2n}\}, n \geq 3,$$

then $\{v_1, v_4\}$ is a perfect transversal of $\mathcal{S}_\alpha(C_{2n})$ and $\{v_1, v_4\}$ is a stable set in C_{2n}.

$f(X)$, for all $X \in \mathcal{W}$; this new function will be denoted also by $f : \mathcal{W} \to \mathcal{Y}$.

Definition 2. *Let G, H be two graphs. A function $f : G \to H$ is called:*

(i) a *homomorphism*, if $f(a)f(b) \in E(H)$ holds for every $ab \in E(G)$;

(ii) a *full homomorphism*, if f is a homomorphism satisfying also

(*) for every $y_1y_2 \in E(H)$ having $f^{-1}(y_i) \neq \emptyset, i = 1, 2$, it follows $f^{-1}(y_1) \sim f^{-1}(y_2)$;

(iii) a *quasi-homomorphism* (or, shortly, a *q-homomorphism*), if for every $ab \in E(G)$, either $f(a) = f(b)$ or $f(a)f(b) \in E(H)$;

(iv) a *full q-homomorphism*, if f is a q-homomorphism that satisfies (*);

(v) a *(q-)epimorphism*, if f is a (q-)homomorphism and a surjection;

(vi) an *isomorphism* whenever f is a bijection and f, f^{-1} are both homomorphisms.

(vii) a graph homomorphism $f : G \to H$ is called faithful if $f(G)$ is an induced subgraph of H i.e. for any $y_1, y_2 \in Im(f)$ with $y_1 \sim y_2$, there are some $x_1, x_2 \in V(G)$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \sim x_2$.

The following results will be used in this paper:

Theorem 1.[5] *Let $f : G \to H$ be a full graph q-epimorphism. Then the following statements are true:*

(i) *it the image by f of any perfect transversal of G is a perfect transversal of H ($f(S) \in \mathcal{T}_s(H)$, for any $S \in \mathcal{T}_s(G)$; $f(Q) \in \mathcal{T}_c(H)$, for any $Q \in \mathcal{T}_c(G)$);*

(ii) *if T is a perfect transversal of H, then any perfect transversal of the induced subgraph by $f^{-1}(T)$ in G is a perfect transversal of G (if $T \in \mathcal{T}_s(H)$ and $S \in \mathcal{T}_s(G[f^{-1}(T)])$, then $S \in \mathcal{T}_s(G)$; if $C \in \mathcal{T}_c(H)$ and $Q \in \mathcal{T}_c(G[f^{-1}(C)])$, then $Q \in \mathcal{T}_c(G)$).*

Theorem 2.[5] *If $f : G \to H$ is a full graph q-epimorphism, T is a perfect transversal of H such that for any $y \in T$, $G[f^{-1}(y)]$ has a perfect transversal T_y, then $\bigcup_{y \in T} T_y$ is a perfect transversal of G.*

2.Perfect transversals in a Composition graph

Definition 3.[8] *Let X be a graph and $(G_x : x \in V(X))$ a family of graphs indexed by the vertex set $V(X)$. The X-join (or the composition graph) of the family of graphs $(G_x : x \in V(X))$ is the graph $G = X[G_x]$ having the vertex* *set* $V(G) = \cup\{\{x\} \times V(G_x) : x \in V(X)\}$ *and the edge set obtained according to the following adjacency rule:*

$$(x, a) \sim (y, b) \Leftrightarrow either (i)\ x \sim y\ or (ii)\ x = y\ and\ a \sim b.$$

EXAMPLE 1.*If* $f : G \to H$ *is a full graph q-epimorphism, then* G *is the* $H - join$ *of the family of the subgraphs induced by* $f^{-1}(y)$, $y \in V(H)$.

A vertex set $A \subset V$ is called a *module* of the graph $G = (V, E)$, if for any $x \in V - A$ either $x \sim A$ or $x \nsim A$.

For instance, the graphs G_x, $x \in V(X)$ are modules of G. Therefore, any full graph $q-$epimorphism arises to a modular decomposition of the graph and any modular decomposition arises to a full graph $q-$epimorphism. In this way, all the results concerning full graph $q-$epimorphisms can be applied to the graph itself.

The modular decomposition of a graph gives a tree representation of the graph by means of modules. Given such a tree representation, it is possible to solve certain combinatorial problems on the graph, by designing an algorithm which derives the solution of the problem from those for the single components induced by the modules of the decomposition tree. This algorithm allows to design the most known efficient algorithms for computing the maximum weight clique, the maximum independent set and the minimum number of cliques necessary to cover all the vertices of a graph, and other combinatorial problems on particular classes of graphs.

These problems are NP-hard for general graphs, but they can be solved in polynomial time for restricted classes of graphs that admit a tree representation satisfying certain properties.

The modular decomposition is an useful tool in many other fields. For example, in biology, it defines the organization of protein-interaction networks as a hierarchy of nested modules [3]. It derives the logical rules of how to combine proteins into the actual functional complexes by identifying groups of proteins acting as a single unit (sub-complexes) and those that can be alternatively exchanged in a set of similar complexes. The method is applied to experimental data on the pro-inflammatory tumor necrosis factor.

Let us recall as an important result concerning the perfect transversals in a composition graph the following theorem due to E. Olaru and E. Mandrescu.

THEOREM 3. [6]*If* G *is the* X*-join of the graph family* $\{G_x : x \in V(X)\}$, *then* G *has a perfect transversal* W, *if and only if* X *has a perfect transversal*

T such that, for every $x \in T$, the graph G_x has a perfect transversal T_x. If so, then $W = \bigcup_{x \in T} T_x$.

We will give another proof of this theorem using graph q-homomorphisms.

Firstly, we present other properties of graph q-homomorphisms.

Let us notice that, if $f : G \to H$ is a graph q-homomorphism, then $f^{-1}(A)$ is not necessarily a stable set (clique), whenever A is stable (a clique) in H. However, the structure of $f^{-1}(A)$ is not completely arbitrary.

Proposition 1.*Let $f : G \to H$ be a full q-epimorphism. If .$Q = \{y_i : 1 \leq i \leq k\}, k \geq 2$, is a clique in H, then for the partition $\{X_i, 1 \leq i \leq k\}, k \geq 2$, with $X_i = f^{-1}(y_i)$, of $X = f^{-1}(Q)$, it holds $X_i \sim X_j$, for all $i, j \in \{1, 2, ..., k\}, i \neq j$, $\mathcal{C}(G[X]) \subseteq \mathcal{C}(G)$, and $f(C) = Q$, for every $C \in \mathcal{C}(G[X])$. Similarly for a stable set of H.*

Proof. Since f is an epimorphism, each $X_i = f^{-1}(y_i), 1 \leq i \leq k$, $k \geq 2$, is non-empty. Further, $X_i \sim X_j, 1 \leq i < j \leq k$, $k \geq 2$, because $y_i \sim y_j$ and f is full. For every $C \in \mathcal{C}(G[X])$, it follows that $C \cap X_i \neq \emptyset$, for each $i \in \{1, 2, ..., k\}$, $k \geq 2$, (because C is maximal), and, consequently, $f(C) = \{y_i : 1 \leq i \leq k\} = Q$.

Assume, on the contrary, that $\mathcal{C}(G[X]) \subsetneq \mathcal{C}(G)$, i.e., there is some $C \in \mathcal{C}(G[X]) - \mathcal{C}(G)$. Hence, there is a vertex $x \in V(G) - X$, such that $\{x\} \sim C$, which implies that $\{f(x)\} \sim f(C)$, i.e., $\{f(x)\} \sim Q$, in contradiction to the choice of $Q \in \mathcal{C}(H)$. Therefore, $\mathcal{C}(G[X]) \subseteq \mathcal{C}(G)$.

It can be proved in a similar way for a stable set of H.

Proof of the Theorem 3.

Let $G = X[G_x]$ be the $X - join$ of the graph family $(G_x : x \in V(X))$. Then the function $f : V(G) \to V(X)$, $f(y) = x$, for every $y \in V(G_x)$ is obviously a full graph q-epimorphism. Let T be a perfect transversal, say a *clique* transversal, of X such that for any $x \in T$ the graph G_x has a *clique* transversal T_x. Then, by Theorem 2, the set $\bigcup_{x \in T} T_x$ is a *clique* transversal of G.

Conversely, if W is a perfect transversal, say a *clique* transversal, of G, then, by Theorem 1, the set $Z = f(W)$ is a *clique* transversal of the graph X, and $W \subseteq f^{-1}(Z) = \bigcup_{z \in Z} V(G_z)$. It follows that $W = \bigcup_{z \in Z} (W \cap V(G_z))$. By Proposition 1 , we have $\mathcal{C}(G[W]) \subseteq \mathcal{C}(G)$. Consequently, $W \cap V(G_z)$ is a *clique* transversal of the graph G_z, for any $z \in Z$, since W is a *clique* transversal of G.

3.Perfect Transversals in a Strong Graph Product

Definition 4.*Let G_1, G_2 be graphs. The strong (or normal) product of the graphs G_1 and G_2 is the graph denoted by $G_1 \boxtimes G_2$ that has the vertex set $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \boxtimes G_2)$ obtained according to the following adjacency rule:*

$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow$ *either* $(x_1 \sim x_2 and y_1 = y_2) or (x_1 = x_2 and y_1 \sim y_2) or (x_1 \sim x_2 \ and y_1 \sim y_2)$.

Let us notice that the *projections* $\pi_1 : V(G_1 \boxtimes G_2) \to V(G_1)$, $\pi_1(x_1, x_2) = x_1$, respectively, $\pi_2 : V(G_1 \boxtimes G_2) \to V(G_2)$, $\pi_2(x_1, x_2) = x_2$ are faithful graph q-epimorphisms.

The family of all cliques of a strong graph product has a precisse structure *see [1], Lemma 2.3*:

Let G_1 and G_2 be two graphs and $G_1 \boxtimes G_2$ be their strong product. Then $\mathcal{C}(G_1 \boxtimes G_2)$ is exactly $\mathcal{C}(G_1) \times \mathcal{C}(G_2)$.

More exactly we have:

Proposition 2.(i) *If Q_1 is a complete set of G_1 and Q_2 is a complete set of G_2, then $Q_1 \times Q_2$ is a complete set of $G_1 \boxtimes G_2$;*

(ii) *C is a clique of $G_1 \boxtimes G_2$ if and only if there are some cliques, Q_1 of G_1 and Q_2 of G_2, such that $C = Q_1 \times Q_2$.*

Corollary 1. $\omega(G_1 \boxtimes G_2) = \omega(G_1) \cdot \omega(G_2)$.

Let us notice that a similar assertion for the family of all stable sets of a graph strong product does not hold, e.g., if $G = C_5$, then it is not difficult to check that $\alpha(G \boxtimes G) > (\alpha(G))^2$.

Theorem 4.*If the graph G_1 has a stable transversal T_1 and the graph G_2 has a stable transversal T_2, then $T_1 \times T_2$ is a stable transversal for their strong product $G_1 \boxtimes G_2$.*

Proof. Let $T_1 \in \mathcal{T}_s(G_1)$ and $T_2 \in \mathcal{T}_s(G_2)$. Then $T_1 \in \mathcal{S}(G_1)$ and $T_2 \in \mathcal{S}(G_2)$.

We prove that $T_1 \times T_2$ is an independent set in the graph $G_1 \boxtimes G_2$.

Suppose, on the contrary, that there are some $(x_1, x_2), (y_1, y_2) \in T_1 \times T_2$ with $(x_1, x_2) \sim (y_1, y_2)$. Obviously, $x_1, y_1 \in T_1$ and $x_2, y_2 \in T_2$. According to the definition of $G_1 \boxtimes G_2$, it follows that either $x_1 \sim y_1$, which contradicts the fact that T_1 is a stable set in G_1, or $x_2 \sim y_2$, which contradicts the fact that T_2 is a stable set in G_2.

Let Q be a clique of $G_1 \boxtimes G_2$. By Proposition 2, it follows that $Q = Q_1 \times Q_2$, where $Q_1 \in \mathcal{C}(G)$ and $Q_2 \in \mathcal{C}(H)$.

Since T_1 is a stable transversal of the graph G, it results that $|T_1 \cap Q_1| = 1$. Similarly, we get that $|T_2 \cap Q_2| = 1$. Let us denote $T_1 \cap Q_1 = \{x_0\}$ and $T_2 \cap Q_2 = \{y_0\}$. Then, $(x_0, y_0) \in (T_1 \times T_2) \cap (Q_1 \times Q_2) = (T_1 \times T_2) \cap Q$, which ensure that $|(T_1 \times T_2) \cap Q| \leq 1$. Since $T_1 \times T_2$ is an independent set and Q is clique of the graph $G_1 \boxtimes G_2$, it follows that $|(T_1 \times T_2) \cap Q| = 1$. Consequently, $T_1 \times T_2$ is a stable transversal of the graph $G_1 \boxtimes G_2$.

4.Graph Homomorphisms and Graph Perfectness

Using the concept of perfect transversal, we give the following alternative definition of graph perfectness.

Definition 5.(i) *A graph G is called s-perfect (c-perfect, respectively) if for every induced subgraph H of G, the family $\mathcal{S}(H)$ ($\mathcal{C}(H)$, respectively) has a perfect transversal.*

(ii) *A graph G is called perfect if for any induced subgraph H of G, the family $\mathcal{S}_\alpha(H)$ has a perfect transversal, that induces a complete subgraph (or, equivalently, for any induced subgraph H of G, the family $\mathcal{C}_\omega(H)$ has a perfect transversal that is a stable set).*

Let us notice that c-perfect graphs are known under the name of *strongly perfect* graphs and were defined by Berge and Duchet in [2]. Clearly, a graph G is c-perfect if and only if its complement is s-perfect, and any c-perfect or s-perfect graph is also perfect. Unlike the case of perfect graphs, the c-perfectness of a graph does not imply the s-perfectness of the same graph, and vice-versa. For instance, $K_n, n \geq 1$, is both c-perfect and s-perfect, while $C_{2n}, n \geq 3$, is only c-perfect.

The results on graph homomorphisms that we obtain allow us to infer some results concerning perfect graphs.

Proposition 3.*If $f : G \to H$ is a full graph epimorphism, then the following assertions are true:*

(i) *$f(Q) \in \mathcal{C}_\omega(H)$, for any $Q \in \mathcal{C}_\omega(G)$;*

(ii) *$f : \mathcal{C}_\omega(G) \to \mathcal{C}_\omega(H)$ is surjective;*

(iii) *if G has a stable transversal for $\mathcal{C}_\omega(G)$, then H has a stable transversal for $\mathcal{C}_\omega(H)$;*

(iv) *if G is perfect, then H is perfect, too.*

Proposition 4.*Let $f : G \to H$ be a full graph q-epimorphism. Then the following assertions are true:*

(i) *if G is c-perfect, then H is c-perfect, too;*

(ii) *if G is s-perfect, then H is s-perfect, too;*

(iii) *if G is (c, s)-perfect, then H is (c, s)-perfect, too.*

Proposition 5.*Let $f : G \to H$ be a full graph epimorphism. Then the following assertions are true:*

(i) *G is c-perfect if and only if H is c-perfect;*

(ii) *G is s-perfect if and only if H is s-perfect;*

(iii) *G is (c, s)-perfect if and only if H is (c, s)-perfect.*

Corollary 2.*If some vertices of a c-,s-,(c, s)-perfect graph G are substituted by perfect graphs of the type of G, then a graph having the same perfectness property is obtained.*

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