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ON AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

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Abstract.Existence and uniqueness of the solution for a class of integral equations with modified argument is proved, using the Contractions Principle.

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1.Introduction

In the paper [4] have been studied for the integral equation

$$x(t)=\int_{-t}^{t}K(t,s,x(s))ds+f(t),\ \ t\in[-T,T],\ \ T>0 \tag{1}$$

the existence and uniqueness of the solution in $C[-T,T]$.

We shall study the integral equation with modified argument

$$x(t)=\int_{-t}^{t}K(t,s,x(s),x(g(s)))ds+f(t),\ \ t\in[-T,T],\ \ T>0 \tag{2}$$

and we shall establish two results concerning the existence and uniqueness of the solution of this equation in $C[-T,T]$ and in the $\overline{B}(f;R)$ sphere, using Contractions Principle (see [2]).

2.Existence of the solution

The results from this section have been established by consulting the papers [1], [2], [3], [4].

A. Existence of the solution in $C[-T,T]$

Let now the integral equation with modified argument (2) and assume that the following conditions are satisfied:

(v_1) $K \in C([-T,T] \times [-T,T] \times \mathbf{R}^2)$;
(v_2) $f \in C[-T,T]$;
(v_3) $g \in C\,([-T,T],\,[-T,T])$.
In addition we suppose that
(v_4) There exists a function $L : \mathbf{R} \to \mathbf{R}_+^*$ such that

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leq L(s)\left(|u_1 - u_2| + |v_1 - v_2|\right),$$

for all $t, s \in [-T,T]$, $u_1, u_2, v_1, v_2 \in \mathbf{R}$, and
(v_5) $q := \int_{-T}^{T} L(s)ds < \frac{1}{2}$.

In these conditions we have the following result concerning the existence and uniqueness of the solution of the integral equation (2) in $C[-T,T]$:

THEOREM 2.1. *Suppose (v_1)-(v_5) are satisfied. Then*
(a) the integral equation (2) has a unique solution $x^* \in C[-T,T]$;
(b) for all $x_0 \in C[-T,T]$, the sequence $(x_n)_{n\in\mathbf{N}}$ defined by the relation

$$x_{n+1}(t) = \int_{-t}^{t} K(t,s,x_n(s),x_n(g(s)))ds + f(t)\,,\ \ n \in \mathbf{N} \tag{3}$$

converges uniformly to the solution x^* .

Proof. We attach to the integral equation (2) the operator $A : C[-T,T] \to C[-T,T]$, defined by

$$A(x)(t) := \int_{-t}^{t} K(t,s,x(s),x(g(s)))ds + f(t),\ \ t \in [-T,T]\,. \tag{4}$$

The set of the solutions of the integral equation (2) coincide with the set of fixed points of the operator A.

We assure the conditions of the *Contractions Principle* and therefore, the operator A must be a contraction. By (v_4) we have

$$|A(x_1)(t) - A(x_2)(t)| \leq \int_{-t}^{t} |K(t,s,x_1(s),x_1(g(s))) - K(t,s,x_2(s),x_2(g(s)))|\,ds \leq$$

$$\leq \int_{-t}^{t} L(s)\left(|x_1(s) - x_2(s)| + |x_1(g(s)) - x_2(g(s))|\right)ds$$

for all $x_1,\, x_2 \in C[-T,T]$ and $t \in [-T,T]$.

Now using the Chebyshev norm, we have

$$\|A(x_1) - A(x_2)\|_{C[-T,T]} \leq 2q \|x_1 - x_2\|_{C[-T,T]} \ , \tag{5}$$

where

$$q := \int_{-T}^{T} L(s)ds \, .$$

Therefore, by (v_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 2q$. The proof result from the *Contractions Principle.*

B. Existence of the solution in the $\overline{B}(f;R)$ sphere

We suppose the following conditions are satisfied:
(v_1') $K \in C([-T,T] \times [-T,T] \times J^2)$, $J \subset \mathbf{R}$ closed interval;
(v_4') There exists a function $L : \mathbf{R} \to \mathbf{R}_+^*$ such that

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leq L(s) \left(|u_1 - u_2| + |v_1 - v_2| \right),$$

for all $t, s \in [-T,T]$, $u_1, u_2, v_1, v_2 \in J$ and also the conditions (v_2), (v_3), (v_5) are satisfied.

We denote M_K a positive constant such that, for the restriction $K|_{[-T,T]\times[-T,T]\times J^2}$, $J \subset \mathbf{R}$ compact, we have

$$|K(t,s,u,v)| \leq M_K, \quad for \ all \ \ t, \, s \in [-T,T], \ u,v \in J \tag{6}$$

and we suppose that the following condition is satisfied:
(v_6) $2TM_K \leq R$ (the invariability condition of the $\overline{B}(f;R)$ sphere).

In these conditions we have the following result of the existence and uniqueness of the solution of the integral equation (2) in the $\overline{B}(f;R)$ sphere:

Theorem 2.2. *Suppose (v_1'), (v_2), (v_3), (v_4'), (v_5) and (v_6) are satisfied. Then*

(a) the integral equation (2) has a unique solution $x^* \in \overline{B}(f;R) \subset C[-T,T]$;

(b) for all $x_0 \in \overline{B}(f;R) \subset C[-T,T]$, the sequence $(x_n)_{n\in\mathbf{N}}$ defined by the relation (3) converges uniformly to the solution x^*.

Proof. We attach to the integral equation (2) the operator $A : \overline{B}(f;R) \to C[-T,T]$, defined by the relation (4), where R is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;R)\right] \Longrightarrow [x(t) \in J \subset \mathbf{R}]$$

and we suppose that there exists at least one number R with this property.

We establish under what conditions, the $\overline{B}\,(f;R)$ sphere is an invariant set for the operator A . We have

$$|A(x)(t) - f(t)| = \left|\int_{-t}^{t} K(t,s,x(s),x(g(s)))ds\right| \leq \int_{-t}^{t} |K(t,s,x(s),x(g(s)))|\, ds$$

and by (6) we have

$$|A(x)(t) - f(t)| \leq 2TM_K, \quad for\ \ all\ \ t \in [-T,T]$$

and then by (v_6) it results that the $\overline{B}(f;R)$ sphere is an invariant set for the operator A. Now we have the operator $A : \overline{B}(f;R) \to \overline{B}(f;R)$, also noted with A, defined by same relation, where $\overline{B}(f;R)$ is a closed subset of the Banach space $C[-T,T]$.

The set of the solutions of the integral equation (2) coincide with the set of fixed points of the operator A .

By a similar reasoning as in the proof of theorem 2.1 and using the conditions (v_4') and (v_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 2q$. Therefore the conditions of the *Contractions Principle* are hold, it results that the operator A has a unique fixed point, $x^* \in \overline{B}(f;R)$ and consequently, the integral equation (2) has a unique solution $x^* \in \overline{B}(f;R) \subset C[-T,T]$. The proof is complete.

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