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SOME OBSERVATIONS ON A CLASS OF D-LINEAR CONNECTIONS ON THE TANGENT BUNDLE

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Abstract.The present paper deals with some problems on the conformal structure on TM. The concepts of conformal structure and d-linear connection compatible with the conformal structure, corresponding to two 1-forms are introduced in the tangent bundle. The problem of determining the set of all d-linear connections compatible with the conformal structure, corresponding to two 1-forms is solved for an arbitrary nonlinear connection.Some important particular cases are considered.

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1. Preliminaries

The geometry of the tangent bundle (TM, π, M) has been studied by M. Matsumoto in [4], by R. Miron and M. Anastasiei in [5], [6], by R. Miron and M. Hashiguchi in [7], by V. Oproiu in [8], by Gh. Atanasiu and I. Ghinea in [1], by R. Bowman in [2], by K. Yano and S. Ishihara in [10],etc. Concerning the terminology and notations, we use those from [6].

Let M be a real n-dimensional C$^{\infty}$-differentiable manifold and (TM, π, M) its tangent bundle.

If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordinates on $\pi^{-1}(U)$ is $(x^i, y^i), (i = 1, ..., n)$.

Let N be a nonlinear connection on TM, with the coefficients $N^i{}_j(x, y), (i, j = 1, ..., n)$.

2. The notion of d- linear connection compatible with a conformal structure

We consider on TM a metrical (almost symplectic) structure G defined by:

$$G(x,y)=\frac{1}{2}g_{ij}(x,y)dx^i\wedge dx^j+\frac{1}{2}\tilde{g}_{ij}(x,y)\delta y^i\wedge\delta y^j, \qquad (1)$$

where $(dx^i,\delta y^i),(i=1,...,n)$ is the dual basis of $\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$, and $(g_{ij}(x,y)$, $\tilde{g}_{ij}(x,y))$ is a pair of given d-tensor fields on TM, of the type (0,2), each of them nondegenerate and symmetric (alternate), in the last case it is necessary that n=2n'.

We asociate to the lift G the Obata's operators:

$$\begin{cases}\Omega^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j-g_{sj}g^{ir}),\ \Omega^{*ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j+g_{sj}g^{ir}),\\ \tilde{\Omega}^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j-\tilde{g}_{sj}\tilde{g}^{ir}),\ \tilde{\Omega}^{*ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j+\tilde{g}_{sj}\tilde{g}^{ir}).\end{cases} \qquad (2)$$

Obata's operators have the same properties as the ones associated with a Finsler space [7].

Let $\mathcal{S}_2(TM)$ be the set of all symmetric d-tensor fields, of the type (0,2) on TM ($\mathcal{A}_2(TM)$ be the set of all alternate d-tensor fields, of the type (0,2) on TM). As is easily shown, the relations on $\mathcal{S}_2(TM)$ ($\mathcal{A}_2(TM)$) defined by (3):

$$\begin{cases}(a_{ij}\sim b_{ij})\Leftrightarrow\left((\exists)\,\lambda(x,y)\in\mathcal{F}(TM),a_{ij}(x,y)=e^{2\lambda(x,y)}b_{ij}(x,y)\right),\\ \left(\tilde{a}_{ij}\sim\tilde{b}_{ij}\right)\Leftrightarrow\left((\exists)\,\mu(x,y)\in\mathcal{F}(TM),\tilde{a}_{ij}(x,y)=e^{2\mu(x,y)}\tilde{b}_{ij}(x,y)\right),\end{cases} \qquad (3)$$

is an equivalence relation on $\mathcal{S}_2(TM)$ ($\mathcal{A}_2(TM)$).

Theorem 1. *The equivalent class: $\hat{G}$ of $\mathcal{S}_2(TM)/_\sim$ ($\mathcal{A}_2(TM)/_\sim$) to which the metrical (almost symplectic) tensor field G belongs, is called conformal structure on TM.*

Thus:

$$\hat{G}=\{G'|G'_{ij}(x,y)=e^{2\lambda(x,y)}g_{ij}(x,y)\ and\ \tilde{G}'_{ij}(x,y)=e^{2\mu(x,y)}\tilde{g}_{ij}(x,y)\}. \qquad (4)$$

Definition 1. *A d-linear connection, D, on TM, with local coefficients $D\Gamma(N)=(L^i_{\ jk},\tilde{L}^i_{\ jk},\tilde{C}^i_{\ jk},C^i_{\ jk})$, for which there exists the 1-forms ω and $\tilde{\omega}$ on*

TM:
$\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i,\ \tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ *such that:*

$$\begin{cases} g_{ij|k} = 2\omega_k g_{ij}, & g_{ij}|_k = 2\dot{\omega}_k g_{ij}, \\ \tilde{g}_{ij|k} = 2\tilde{\omega}_k \tilde{g}_{ij}, & \tilde{g}_{ij}|_k = 2\ \dot{\tilde{\omega}}_k\ \tilde{g}_{ij}, \end{cases} \tag{5}$$

where $|$ *and* $\big|$ *denote the h-and v-covariant derivatives with respect to D,is said to be compatible with the conformal structure $\hat{G}$, corresponding to the 1-forms $\omega, \tilde{\omega}$ and is denoted by: $D\Gamma(N, \omega, \tilde{\omega})$.*

3. The set of all d- linear connections compatible with the conformal structure Ĝ, corresponding to two 1-forms

Let $\overset{0}{N}$ and N be two nonlinear connections on TM, with the coefficients $(\overset{0}{N}_{(1)}{}^i{}_j, \overset{0}{N}_{(2)}{}^i{}_j)$ and $(N_{(1)}{}^i{}_j, N_{(2)}{}^i{}_j)$ respectively.

Let $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L}{}^i{}_{jk}, \overset{0}{\tilde{L}}{}^i{}_{jk}, \overset{0}{\tilde{C}}{}^i{}_{jk}, \overset{0}{C}{}^i{}_{jk})$ be the local coefficients of a fixed d-linear connection $\overset{0}{D}$ on TM. Then any d-linear connection, D, on TM, with local coefficients: $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$,can be expresed in the form:

$$\begin{cases} N^i{}_j = \overset{0}{N}{}^i{}_j - A^i{}_j, \\ L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + A^l{}_k\ \overset{0}{\tilde{C}}{}^i{}_{jl} - B^i{}_{jk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + A^l{}_k\ \overset{0}{C}{}^i{}_{jl} - \tilde{B}^i{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} - \tilde{D}^i{}_{jk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} - D^i{}_{jk}, \\ \\ A^l{}_{\overset{0}{j|k}} = 0, \end{cases} \tag{6}$$

where $(A^i{}_j, B^i{}_{jk}, \tilde{B}^i{}_{jk}, \tilde{D}^i{}_{jk}, D^i{}_{jk})$ are components of the difference tensor fields of $D\Gamma(N)$ from $\overset{0}{D}\ \Gamma(\overset{0}{N})$, [4] and $\overset{0}{|}$, $\overset{0}{\big|}$ denotes the h-and v-covariant derivatives with respect to $\overset{0}{D}$.

Using a well known method given by R.Miron in [5] for the case of the Finsler connections we obtain:

Theorem 2.*Let* $\overset{0}{D}$ *be a given d-linear connection on* TM*, with local coefficients* $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$*. Then set of all d-linear connections compatible with the conformal structure* $\hat{G}$*, corresponding to the 1-forms* ω *and* $\tilde{\omega}$*, with local coefficients* $D\Gamma(N, \omega, \tilde{\omega}) = ({L^i}_{jk}, {\tilde{L}^i}_{jk}, {\tilde{C}^i}_{jk}, {C^i}_{jk})$ *is given by:*

$$\begin{cases} N^i_j = \overset{0}{N^i_j} - X^i_j, \\ {L^i}_{jk} = \overset{0}{L^i}_{jk} + \overset{0}{\tilde{C}^i}_{jm} {X^m}_k + \frac{1}{2} g^{is}(g_{sj\overset{0}{|}k} + g_{sj} \overset{0}{|}_m {X^m}_k) - {\delta^i}_j \omega_k + \Omega^{ir}_{hj} {X^h}_{rk}, \\ {\tilde{L}^i}_{jk} = \overset{0}{\tilde{L}^i}_{jk} + \overset{0}{C^i}_{jm} {X^m}_k + \frac{1}{2} \tilde{g}^{is}(\tilde{g}_{sj\overset{0}{|}k} + \tilde{g}_{sj} \overset{0}{|}_m {X^m}_k) - {\delta^i}_j \tilde{\omega}_k + \tilde{\Omega}^{ir}_{hj} {\tilde{X}^h}_{rk}, \\ {\tilde{C}^i}_{jk} = \overset{0}{\tilde{C}^i}_{jk} + \frac{1}{2} g^{is} g_{sj} \overset{0}{|}_k - {\delta^i}_j \dot{\omega}_k + \Omega^{ir}_{hj} {\tilde{Y}^h}_{rk}, \\ {C^i}_{jk} = \overset{0}{C^i}_{jk} + \frac{1}{2} \tilde{g}^{is} \tilde{g}_{sj} \overset{0}{|}_k - {\delta^i}_j \dot{\tilde{\omega}}_k + \tilde{\Omega}^{ir}_{hj} Y^h_{rk}, \\ {X^i}_{j\overset{0}{|}k} = 0, \end{cases} \tag{7}$$

where X^i_j, ${X^i}_{jk}$, ${\tilde{X}^i}_{jk}$, ${\tilde{Y}^i}_{jk}$, Y^i_{jk} are arbitrary tensor fields on TM, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are arbitrary 1-forms in TM and $\overset{0}{|}$, $\overset{0}{\big|}$ denote the h-and respective v-covariant derivatives with respect to $\overset{0}{D}$.

Particular cases

1. If ${X^i}_j = {X^i}_{jk} = {\tilde{X}^i}_{jk} = {\tilde{Y}^i}_{jk} = {Y^i}_{jk} = 0$ in Theorem 2, we have:

Theorem 3.*Let* $\overset{0}{D}$ *be a given d-linear connection on* TM*, with local coefficients* $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$*. Then the following d-linear connection* D*, with local coefficients* $D\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = ({L^i}_{jk}, {\tilde{L}^i}_{jk}, {\tilde{C}^i}_{jk}, {C^i}_{jk})$ *given by (8) is compatible with the conformal structure* $\hat{G}$*, corresponding to the 1-forms* ω *and* $\tilde{\omega}$*:*

$$\begin{cases} L^i{}_{jk} = \overset{0}{L^i}{}_{jk} + \frac{1}{2} g^{is} g_{\underset{sj|k}{0}} - \delta^i_j \omega_k, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}{}_{jk} + \frac{1}{2} \tilde{g}^{is} \tilde{g}_{\underset{sj|k}{0}} - \delta^i_j \tilde{\omega}_k, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}{}_{jk} + \frac{1}{2} g^{is} g_{sj} \overset{0}{|}_k - \delta^i_j \dot{\omega}_k, \\ C^i{}_{jk} = \overset{0}{C^i}{}_{jk} + \frac{1}{2} \tilde{g}^{is} \tilde{g}_{sj} \overset{0}{|}_k - \delta^i_j \dot{\tilde{\omega}}_k, \end{cases} \tag{8}$$

where $\overset{0}{|}$, $\overset{0}{\Big|}$ denote the h-and respective v-covariant derivatives with respect to the given d-linear connection $\overset{0}{D}$ and $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are two given 1-forms in TM.

2. If we take a metrical (almost symplectic) d-linear connection as $\overset{0}{D}$ in Theorem 3, then (8) becomes:

$$\begin{cases} L^i{}_{jk} = \overset{0}{L^i}{}_{jk} - \delta^i_j \omega_k, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}{}_{jk} - \delta^i_j \tilde{\omega}_k, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}{}_{jk} - \delta^i_j \dot{\omega}_k, \\ C^i{}_{jk} = \overset{0}{C^i}{}_{jk} - \delta^i_j \dot{\tilde{\omega}}_k. \end{cases} \tag{9}$$

3. If we consider a d-linear connection compatible with conformal structure $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$, as $\overset{0}{D}$ in Theorem 2, we have

Theorem 4.*Let $\overset{0}{D}$ be a given d-linear connection compatible with conformal structure $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$ on TM, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (\overset{0}{L^i}{}_{jk}, \overset{0}{\tilde{L}^i}{}_{jk}, \overset{0}{\tilde{C}^i}{}_{jk}, \overset{0}{C^i}{}_{jk})$. The set of all d-linear connections compatible with conformal structure $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$ on TM, with local coefficients $D\Gamma(N, \omega, \tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N^i_{\ j}} - X^i_{\ j}, \\ L^i_{\ jk} = \overset{0}{L^i_{\ jk}} + (\overset{0}{\tilde{C}^i_{\ jm}} + \delta^i_j \dot{\omega}_m) X^m_{\ k} + + \Omega^{ir}_{hj} X^h_{\ rk}, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i_{\ jk}} + (\overset{0}{C^i_{\ jm}} + \delta^i_j \dot{\tilde{\omega}}_m) X^m_{\ k} + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h_{\ rk}, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i_{\ jk}} + \Omega^{ir}_{hj} \tilde{Y}^h_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C^i_{\ jk}} + \tilde{\Omega}^{ir}_{hj} Y^h_{\ rk}, \\ X^i_{\ j \overset{0}{|} k} = 0, \end{cases} \tag{10}$$

where $X^i_{\ j}$, $X^i_{\ jk}$, $\tilde{X}^i_{\ jk}$, $\tilde{Y}^i_{\ jk}$, $Y^i_{\ jk}$ are arbitrary tensor fields on TM, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are two arbitrary 1-forms in TM and $\overset{0}{|}$, $\overset{0}{\Big|}$ denote h-and respective v-covariant derivatives with respect to $\overset{0}{D}$.

The result obtained in this particular case support the findings of R.Miron and M.Hashiguchi for the Finsler connections in their paper [7].

4. If we take $X^i_{\ j} = 0$ in Theorem 3 we obtain a result given by R.Miron and M.Anastasiei in [6]:

Theorem 5.*Let* $\overset{0}{D}$ *be a given d-linear connection compatible with conformal structure* $\hat{G}$*, corresponding to the 1-forms* ω *and* $\tilde{\omega}$ *on* TM*, with local coefficients:* $\overset{0}{D}\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (\overset{0}{L^i_{\ jk}}, \overset{0}{\tilde{L}^i_{\ jk}}, \overset{0}{\tilde{C}^i_{\ jk}}, \overset{0}{C^i_{\ jk}})$*. The set of all d-linear connections compatible with conformal structure* $\hat{G}$*, which preserve the nonlinear connection* $\overset{0}{N}$*, corresponding to the 1-forms* ω *and* $\tilde{\omega}$ *on* TM*, with local coefficients* $D\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ *is given by:*

$$\begin{cases} L^i{}_{jk} = \overset{0}{L^i}{}_{jk} + \Omega^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}{}_{jk} + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}{}_{jk} + \Omega^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C^i}{}_{jk} + \tilde{\Omega}^{ir}_{hj} Y^h_{rk}, \end{cases} \tag{11}$$

where $X^i_{\ j}$, $X^i{}_{jk}$, $\tilde{X}^i_{jk}$, $\tilde{Y}^i_{jk}$, Y^i_{jk} are arbitrary tensor fields on TM.

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