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THE CYLINDRICAL TRANSFORM $D_C(R,Z)$ OF THE FUNCTIONS SPACE $D(R^3)$

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Abstract. We define the space of test function $D_c(r,z)$ and we give some of its properties. The cylindrical transformation is $T_c : D(R^3) \longrightarrow D_c(r,z) \subset D(R^2)$ defined and it is shown that T_c is a linear and continuous operator from $D(R^3)$ in $D_c(r,z)$.

Key words: distributions, cylindrical transform

1. Introduction

To solve certain problems from the physical-mathematics, sometimes it is useful to change the Cartesian coordinates $(x,y,z) \in R^3$ into the cylindrical coordinates $(r,\theta,z) \in R^3$. This necessity leads to the writing of the distributions in the cylindrical coordinates, for which we shall define the function test space $D_c(r,z)$ as well as the cylindrical transform T_c. Both for the test space $D_c(r,z)$ in the cylindrical coordinates (r,z) and for the cylindrical transform T_c associated, certain properties are established. These allow the study of some distributions representable only with respect to the cylindrical coordinates $(r,z) \subset R^2$.

2. General results

Let be the application $T : R^3 \longrightarrow R^3$ defined by the relations:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \tag{1}$$

these relations define the univocal transformation from the cylindrical coordinates $(r,\theta,z) \in R^3$ to the Cartesian coordinates $(x,y,z) \in R^3$, having the Jacobian of the transform

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

If to the punctual transformation (1) we impose the restrictions $r \geq 0, \theta \in [0, 2\pi), z \in R$, then the transformation (1) becomes locally bijective everywhere with the exception of the points $(0,0,z_0) \in R^3$. Thus, the origin O of the Cartesian system of coordinates represents a singular point, because $J(r,\theta,z) = r = 0$, which in the cylindrical coordinates is defined by $r = 0, z = 0$, and $\theta \in R$ arbitrary.

To the punctual transformation T defined by (1) we associate the functions space $D_c(r,z)$.

Definition 2.1. *We call space of the test functions $D_c(r,z)$, the set of the functions*

$$D_c(r,z) = \left\{ \psi \mid \ \psi : R^2 \longrightarrow R, \psi(r,z) = \int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta, z)d\theta, \varphi \in D(R^3) \right\}. \tag{2}$$

Proposition 2.1. *The space $D_c(r,z)$ has the properties*

1. $D_c(r,z) \subset D(R^2)$;

2. The function $\psi(r,z)$ is an even function with respect to $r \in R$ and

$$\frac{\partial^k \psi(0,0)}{\partial r^k} = \begin{cases} 0 \ , \quad \text{for } k \text{ odd} \\ \sum_{n+m=k} \dfrac{k!}{n!m!} a_{nm} \dfrac{\partial^{n+m}\varphi(0,0,0)}{\partial x^n \varphi y^m} \ , \quad \text{for } k \text{ even} \end{cases} \tag{3}$$

where a_{nm} has the expression

$a_{nm} = \int_0^{2\pi} \cos^n\theta \sin^m\theta d\theta =$

$$= \begin{cases} 0 \ , \quad \text{for } m \text{ and } n \text{ odds or } m+n \text{ odd} \\ \dfrac{1\cdot 3\cdot 5\cdot ...\cdot(n-1)}{(m+1)(m+3)...(m+n-1)} \cdot \dfrac{\pi}{2^{m+n-1}} \cdot \dfrac{(m+n)!}{\left[\left(\dfrac{m*n}{2}\right)!\right]^2}, \text{for } m \text{ and } n \text{ evens} \end{cases} \tag{4}$$

Proof. The function $\varphi^*(r,\theta,z) = \varphi(r\cos\theta, r\sin\theta, z)$ is obviously indefinite derivable, resulting from the composition of functions of the same class. Because the function $\varphi \in D(R^3)$ is with compact support, results that for $|r|$ or $|z|$ big enough, tends to 0 and thus we deduce that ψ is with compact support, then $\psi(r,z) \in D(R^2)$, i.e. $D_c(r,z) \subset D(R^2)$.

To justify the property 2 we shall show that the function $\psi(r,z)$ is an even function with respect to $r \in R$. With that end in view we observe that the function

$$\varphi^*(r,\theta,z) = \varphi(r\cos\theta, r\sin\theta, z) \ ; \ (r,\theta,z) \in R^3$$

is a periodic function with the period 2π with respect to the variable $\theta \in R$.
Then, we can write

$$\psi(r,z) = \int_0^{2\pi} \varphi^*(r,\theta,z)d\theta = \int_0^{a+2\pi} \varphi^*(r,\theta,z)d\theta, a \in R. \qquad (5)$$

Making the change of variable $u = \theta - \pi$ we obtain
$\psi(r,z) = \int_{-\pi}^{\pi} \varphi^*(r,u+\pi,z)d\theta = \int_{-\pi}^{\pi} \varphi(r\cos(u+\pi), r\sin(u+\pi), z)du = \int_{-\pi}^{\pi} \varphi(-r\cos u, -r\sin u, z)du = \int_{-\pi}^{\pi} \varphi^*(-r,\theta,z)d\theta = \int_0^{2\pi} \varphi^*(-r,\theta,z)d\theta = \psi(-r,z)$, wherefrom results that $\psi(r,z)$ is an even function with respect to the variable $r \in R$.

Because $\psi(r,z)$ is an even function with respect to the variable $a \in R$, we obtain
$\dfrac{\partial^k \psi(0,0)}{\partial r^k} = 0$ for k an odd number.
Differentiating (5) we obtain

$$\frac{\partial^k \psi(r,z)}{\partial r^k} = \int_0^{2\pi} \left(\frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial x}\sin\theta \right)^{(k)} \varphi(r\cos\theta, r\sin\theta, z)d\theta, \qquad (6)$$

wherefrom for $r \longrightarrow 0$ and $z \longrightarrow 0$ we have

$$\frac{\partial^k \psi(0,0)}{\partial r^k} = \int_0^{2\pi} \sum_{\alpha=0}^{k} C_k^\alpha \frac{\partial^k \varphi(0,0,0)}{\partial x^{k-\alpha}\partial y^\alpha} \cos^{k-\alpha}\theta \sin^\alpha \theta d\theta =$$
$$\sum_{n+m=k} \frac{k!}{n!m!} a_{nm} \frac{\partial^{n+m}\varphi(0,0,0)}{\partial x^n \varphi y^m}$$

where a_{nm} has the expression (4).

The expression of the coefficients a_{nm}, it results using the recurrence relation

$$a_{nm} = \frac{n-1}{m+1} a_{n-2\ m+2} \ n, m \in N_0;$$

as well as the formula

$$\int_0^{2\pi} \cos^m \theta d\theta = \int_0^{2\pi} \sin^m \theta d\theta = \begin{cases} 0\ , & \text{for } m \text{ odd} \\ \dfrac{\pi}{2^{m-1}} \cdot \dfrac{m!}{\left[\left(\dfrac{m}{2}\right)!\right]^2}\ , & \text{for } m \text{ even} \end{cases} .$$

With this the proposition is proved.

Obviously, these result shows that the space $D_c(r, z)$ is a subspace of $D(R^2)$.

Definition 2.2. *We call the cylindrical transformation the application*

$$D_c : D(R^3) \longrightarrow D_c(r, z) \subset D(R^2)$$

defined by the relation

$$D_c(\varphi)(r, z) = \psi(r, z)\ ,\ \psi(r, z) = \int_0^{2\pi} \varphi(r \cos\theta, r \sin\theta, z) d\theta, (r, z) \in R^2, \quad (7)$$

where $\varphi \in D(R^3)$*.*
The function

$$\psi(r, z) = D_c(\varphi)(r, z)\ ,$$

represents the cylindrical transformation of the function $\varphi \in D(R^3)$*, and* $D_c(r, z)$ *the cylindrical transform of the space* $D(R^3)$*.*
Obviously, the cylindrical transform T_c is a linear operator.

Proposition 2.2. *Let be* $\varphi \in D(R^3)$ *and* $\psi = T_c(\varphi)$ *. Then it holds the relation*

$$\frac{\partial^2 \psi(0, 0)}{\partial r^2} = \pi \Delta \varphi(0, 0, 0) \quad (8)$$

where Δ *is the Laplace operator in* R^2*, namely:*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. Considering $k = 2$ that in the formula (3) we obtain

$$\frac{\partial^2\psi(0,0)}{\partial r^2} = \sum\nolimits_{n+m=2}\frac{2!}{n!m!}a_{nm}\frac{\partial^2\varphi(0,0,0)}{\partial x^n\varphi y^m} =$$

$$= a_{02}\frac{\partial^2\varphi(0,0,0)}{\varphi y^2} + a_{20}\frac{\partial^2\varphi(0,0,0)}{\partial x^2} + a_{11}\frac{\partial^2\varphi(0,0,0)}{\partial x\varphi y}. \tag{9}$$

Using the formula (4) we shall obtain for the coefficients a_{02}, a_{20}, a_{11} the values

$$a_{02} = \int_0^{2\pi}\sin^2\theta d\theta = \pi,\ \ a_{20} = \int_0^{2\pi}\cos^2\theta d\theta = \pi,\ \ a_{11} = \int_0^{2\pi}\cos\theta\sin\theta d\theta = 0.$$

Substituting these values in (9) we obtain (8).

Concerning the convergence in the spaces $D(R^3)$ and $D_c(r,z) \subset D(R^2)$, we have

Proposition 2.3. *The cylindrical transformation T_c is a continuous linear operator from $D(R^3)$ in $D_c(r,z) = T_c(D(R^3)) \subset D(R^2)$, hence $\varphi_k \overset{D(R^3)}{\longrightarrow} \varphi$ implies $\psi_k = T_c(\varphi_k) \overset{D_c(r,z)}{\longrightarrow} \psi = T_c(\varphi)$.*

Proof. Let be $\varphi \in D(R^3)$ and $\psi = T_c(\varphi)$. Denoting with p_m and p^*_m semi-norms on $D(R^3)$ and on $D_c(r,z)$, respectively, we can write

$$p_m(\varphi) = \sup_{|\alpha|\le m,(x,y,z)\in\omega} |D^\alpha\varphi|, \alpha \in N_0^3, m \in N_0, \sup\varphi \subset \omega, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}} \tag{10}$$

where $\omega \in R^3$ is a compact set, and

$$p^*_m(\psi) = \sup_{|\beta|\le m,(r,z)\in\omega_1} |D^{*\beta}\psi|, \beta \in N_0^2, m \in N_0, \sup\psi \subset \omega_1,\ D^{*\beta} = \frac{\partial^{|\beta|}}{\partial r^{\beta_1}\partial z^{\beta_2}},$$

where $\omega_1 \in R^2$ is a compact set.

Because

$$\psi(r,z) = \int_0^{2\pi}\varphi(r\cos\theta, r\sin\theta, z)d\theta\ ,$$

we have

$$D^{*h}\psi = \int_0^{2\pi} \sum_{m=0}^{h_1} \frac{h_1!}{m!(h_1-m)!} \cdot \frac{\partial^{|h|}\varphi(r\cos\theta, r\sin\theta, z)}{\partial x^{h_1-m}\partial y^m \partial z^{h_2}} \cos^{h_1-m}\theta \sin^m \theta d\theta, \tag{11}$$

where $h = (h_1, h_2) \in N_0^2$.

From the above mentioned the relation results

$$\sup_{(r,z)\in\omega_1} |D^{*h}\psi| \leq c_h, \sup_{|\alpha|\leq|h|,(x,y,z)\in\omega} |D^\alpha\varphi|, \alpha \in N_0^3,\ h \in N_0^2, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}} \tag{12}$$

$c_h > 0$ being a constant depending on $h \in N_0^2$.

Taking into account (10), (12) we obtain

$$p_m^*(\psi) = \sup_{|\beta|\leq m,(r,z)\in\omega_1} |D^{*\beta}\psi| \leq c(m) \sup_{|\alpha|\leq m,(x,y,z)\in\omega} |D^\alpha\varphi|, \tag{13}$$

hence

$$p_m^*(\psi) \leq c(m)p_m(\varphi), \tag{14}$$

where $c(m) > 0$ is a constant dependent upon $m \in N_0$.

The relation (14) emphasize the dependence between the semi-norms p_m^* and p_m corresponding to the spaces $D_c(r,z) \subset D(R^2)$ and $D(R^3)$, respectively.

From the inequality (14) we obtain

$$p_m^*(\psi_k - \psi) \leq c(m)p_m(\varphi_k - \varphi), \tag{15}$$

since $\psi_k - \psi = T_c(\varphi_k - \varphi)$.

Because, according to the hypothesis $\varphi_k \overset{D(R^3)}{\longrightarrow} \varphi$, then $\lim_k p_m(\varphi_k - \varphi) = 0$, from (15) results $\lim_k p_m^*(\psi_k - \psi) = 0$, namely $\psi_k \overset{D_c(r,z)}{\longrightarrow} \psi$, which prove the proposition.

These results will be used to the study of a class of distributions from $D'(R^3)$ representable only with respect to the cylindrical coordinates $(r,z) \in R^2$.

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