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SENSITIVITY ANALYSIS OF COSTS IN A TRANSPOSTATION PROBLEM

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1.Introduction

Let's consider m providers (distributors, manufacturers, etc.) $F_1 - F_m$ of an identical product that they possess in quantities $a_1 - a_m$ as well as n consummers or beneficiaries (users) $B_1 - B_n$ whose needs are $b_1 - b_n$. Let us denote by x_{ij} the quantity that is transported (transferred, allocated) from F_i to B_j and with c_{ij} as unit transportation costs from F_i to B_j. The cost of transportation for the quantity x_{ij} from F_i to B_j will equal $c_{ij} \cdot x_{ij}$ and for all the providers and all the benficiaries, the overall cost will equal $f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$ which must be minimised. Obviously, at the level of provider F_i the total quantity provided will equal $\sum_{j=1}^{n} x_{ij} \leq a_i, 1 \leq i \leq m$, and everything that the beneficiary B_j can get is . We search for that transportation (or allocation) option which satisfies all partners F_i şi B_j and leads to a minimum total cost.

Keeping in mind the considerations and example presented, wecan sum up the fact that, in a general form, a linear model of transportation type may be written as follows:

$$\begin{cases} [opt]f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij} \\ \sum_{j=1}^{n} x_{ij} \leq a_i, 1 \leq i \leq m \\ \sum_{i=1}^{m} x_{ij} \geq b_j, 1 \leq j \leq n \\ x_{ij}, \quad i = \overline{1,m}, \quad j = \overline{1,n} \end{cases}$$

where:

– $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$ either represents the total cost of the transportation operation (when $opt = min$), or the total profit of the transportation operation (when $opt = max$).

– a_i represents the available amount of provider F_i, $i = \overline{1,m}$;

– b_j represents the necessary of the beneficiary B_j, $j = \overline{1,n}$;

– c_{ij} represents the unit cost (or unit profit) for the relation $F_i \longrightarrow B_j$;

– x_{ij} represents the transported quantity (allocated, distributed, transferred) from the provider F_i to the beneficiary B_j, $i = \overline{1,m}$, $\quad j = \overline{1,n}$;

– the restriction $\sum_{j=1}^{n} x_{ij} \leq a_i$ shows that the beneficiaries cannot be allocated more than the provider F_i has available;

– the restriction $\sum_{i=1}^{m} x_{ij} \geq b_j$ shows that the beneficiary B_j is allocated from all beneficiaries at least as much as his necessary is;

– $a = \sum_{i=1}^{m} a_i$ represents the available amount (or supply) of all providers and $b = \sum_{j=1}^{m} a_j$ represents the necessary (or demand) of all beneficiaries in case the sums make sense (otherwise, the problem does not make sense).

2.The distributive algorithm

The stages of the distributive algorithm for determining the optimal solution to the transportation problem (1) are:

1) – we determine an initial basic solution;

2) – we calculate the values of dual variables (marginal values) ui, vj as solutions to the equation system $u_i + v_j = c_{ij}$, where the indices i, j are those of basic variables (occupied boxes);

3) – we verify the optimality of the basic solution, that is we check for the inequality $u_i + v_j \leq c_{ij}$ for all indices corresponding to the non-basic variables (free boxes);

4) – if the basic solution is not optimal, then it must be improved;

5) – we take again stages 2), 3), 4) with the new basic solution and we carry on until all the optimality conditions are fulfilled;

6) – we write the optimal solution and we calculate the minimal value of the purpose function.

Remark.

1. *We usually aim to completely satisfy the demand and the available amount; consequently, the restrictions of the problem appear as inequalities. To these we may add other restrictions with various significances.*

2. *If instead of a certain product that can be really transported we consider that $a_1 - a_m$ are producible areas of the same quality, and $B_1 \ldots\ B_n$ are n cultures that need such a land, then the problem of the optimal arrangement of crops is of the previous mathematic model, but it is not a transportation operation. This is one of the reasons we say transportation-type models.*

3.Sensitivity Analysis

Sensitivity analysis is a particular case of parametric programming.

The main objective of sensitivity analysis is to determine the effect of a certain variation of parameters a_i, b_j and c_{ij} on the optimal solution.

When these parameter variations are low and result in small solution differences, we say that the optimal solution is insensitive to changes. The degree of sensitivity can range from no change of the optimal solution, with the only change in the optimal value of the objective function, up to a considerable change of the optimal solution, in the way that the basis from the last table can change, which gives the optimal solution of the initial problem, so that it may have a new optimal solution with other variables and other values, therefore another optimal value of the objective function.

Further on, we will show the way in which sensitivity analysis is realized and its importance for the problems that have an economic content for the coefficients of the objective function.

4.Sensitivity Analysis for the Coefficients of the Objective Function

In a transportation problem, the coefficients of the objective function c_{ij} of decision variables x_{ij}, represent estimations of unit transportation costs, unit benefits,etc. and are considered to be constant. However, even during transportation, there may be variations to the initially estimated values.

The problem in which the coefficients of the objective function depend linearly to a parameter t, $c_{ij} = c_{ij}^0 + tc_{ij}^1$ with c_{ij}^0, c_{ij}^1 constant, is studied by parametrical programming.

Sensitivity analysis of coefficients from the objective function is a particular case of parametrical programming, where one of the coefficients c_{ij} is replaced by $c_{ij} + t$, $i = \overline{1,n}$, $j = \overline{1,m}$, the rest of the data remaining unchanged. Thus, a number of $n \times m$ problems is resolved, problems in which there is only one coefficient from the objective function that varies.

For $t = 0$ we obtain the initial problem.

Sensitivity analysis establishes the intervals in which each of the objective function coefficients can vary, so that the solution to the initial problem remains the same, so the basis from the last table remains unchanged. We can only change the optimal value of the objective function, as the coefficients depend on the parameter or it may remain the same.

Simultaneously the reduced cost results.

In a nondegenerate problem of maximum (minimum) the cost reduced for a variable of non-basic decision is the quantity with which its coefficient of the objective function may be increased (decreased) so that it becomes basic, so as to have a strictly positive value in a new optimal solution. For the decision variables that were basic (> 0), the reduced cost is zero.

The low cost for a non-used activity also represents the quantity by which the profit will decrease, if an activity that has not been used is forced to be basic. For instance, we have $x_{ij} = 0$ and impose $x_{ij} \geq 1$.

In order to realize the sensitivity analysis of coefficients from the objective function, we have to solve nxm problems: in each of them, only one coefficient c_{ij} is replaced by $c_{ij} + t$, $i = 1, 2, ..., n$, $j = 1, 2, \dots, m$ the rest of the data remaining unchanged, where $t \in R$.

In order to solve just one problem of $n \times m$, we proceed as follows:

1) We draw up the distributive table.

2) We determine the optimal solution of the initial problem.

3) We proceed to a re-optimization of the problem, replacing c_{ij} by $c_{ij} + t$.

4) The optimality condition for the box (i, j) is imposed. The result is the interval in which the parameter t may vary, so that the optimal solution to the initial problem stays optimal.

5) We calculate the limits between which the c_{ij} coefficient may vary. For instance, if $t \in [t_1, t_2]$, we draw up one of the following tables:

- the table with upper and lower limits for the coefficient c_i

Lower limit	Original value	Upper limit
$c_{ij} + t_1$	c_{ij}	$c_{ij} + t_2$

-the table with increases and decreases allowed for the coefficient c_{ij}

Original value	Allowed increase	Allowed decrease
c_{ij}	t_2	t_1

6) In the interval for the optimal solution for $c_{ij} \in [c_{ij} + t_1, c_{ij} + t_2]$ the initial problem is the same. The optimal value of the objective function can change or can remain the same. Outside this interval, the solution to the problem will not be optimal.

5. Application

Make the sensitivity analysis in the transportation problem given through the table:

6	4	3	50
2	5	3	30
25	20	35	

Solution:

We solve at first the initial problem. We have:

	$v_1=-1$		$v_2=4$		$v_3=3$			
$u_1=0$	6	1	4	4	3	3	50	15
		·	+	15		35		
$u_2=2$	2	2	5	5	3	4	30	5
		25	-	5	+	·		
	25		20		35		5	
			5					

The solution is not optimal and we form the cycle of the box (2,3).

	$v_1=2$		$v_2=4$		$v_3=3$	
$u_1=0$	6	2	4	4	3	3
		·		20		30
$u_2=0$	2	2	5	4	3	3
		25		·		5

Therefore, the optimal solution is $x_{opt}=\begin{pmatrix} 0 & 20 & 30 \\ 25 & 0 & 5 \end{pmatrix}$, $f_{min}=235$.

We re-optimize the problem:

- $c_{11} \longrightarrow c_{11}+t$

	$v_1=2$		$v_2=4$		$v_3=3$	
$u_1=0$	$6+t$	2	4	4	3	3
		·		20		30
$u_2=0$	2	2	5	4	3	3
		25		·		5

The solution remains optimal if $6+t \geq 2$, that is $t \geq -4$.
$f_{11}(t) = 235$.

- $c_{12} \longrightarrow c_{12} + t$

	$v_1 = 2$		$v_2 = 4+t$		$v_3 = 3$	
$u_1 = 0$	6	2	4+t	4+t	3	3
		·		20		30
$u_2 = 0$	2	2	5	4+t	3	3
		25		·		5

The solution remains optimal if $5 \geq 4+t$, that is $t \leq 1$.
$f_{12}(t) = 235 + 20t$.

- $c_{13} \longrightarrow c_{13} + t$

	$v_1 = 2$		$v_2 = 4$		$v_3 = 3+t$	
$u_1 = 0$	6	2	4	4	$3+t$	$3+t$
		·		20		30
$u_2 = -t$	2	2	5	$4-t$	3	3
		25		·		5

The solution remains optimal if $6 \geq 2+t$ and $5 \geq 4-t$ that is $t \in [-1, 4]$.
$f_{13}(t) = 235 + 30t$.

- $c_{21} \longrightarrow c_{21} + t$

	$v_1 = 2+t$		$v_2 = 4$		$v_3 = 3$	
$u_1 = 0$	6	$2+t$	4	4	3	3
		·		20		30
$u_2 = 0$	$2+t$	$2+t$	5	$4-t$	3	3
		25		·		5

The solution remains optimal if $6 \geq 2+t$, that is $t \leq 4$.
$f_{21}(t) = 235 + 25t$.

- $c_{22} \longrightarrow c_{22} + t$

	$v_1 = 2$		$v_2 = 4$		$v_3 = 3$	
$u_1 = 0$	6	2	4	4	3	3
		$\cdot$		20		30
$u_2 = 0$	2	2	$5 + t$	4	3	3
		25		$\cdot$		5

The solution remains optimal if $5 + t \geq 4$, that is $t \geq -1$.
$f_{22}(t) = 235$.

- $c_{23} \longrightarrow c_{23} + t$

	$v_1 = 2 - t$		$v_2 = 4$		$v_3 = 3$	
$u_1 = 0$	6	$2 - t$	4	4	3	3
		$\cdot$		20		30
$u_2 = t$	2	2	5	$4 + t$	$3 + t$	$3 + t$
		25		$\cdot$		5

The solution remains optimal if $6 \geq 2 - t$ and $5 \geq 4 + t$, that is $t \in [-4, 1]$.
$f_{23}(t) = 235 + 5t$.

The sensitivity analysis shows that the solution to the initial problem remains optimal when the coefficients vary between the limits given in the following table:

Interval for t	Lower limit	Original value	Upper limit
$t \in [-4, \infty)$	2	$c_{11} = 6$	None
$t \in (-\infty, 1]$	None	$c_{12} = 4$	5
$t \in [1, 4]$	2	$c_{13} = 3$	7
$t \in (-\infty, 4]$	None	$c_{21} = 2$	6
$t \in [-1, \infty)$	4	$c_{22} = 5$	None
$t \in [-4, -1]$	-1	$c_{23} = 3$	4

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