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SOME QUALITATIVE FEATURES OF TURBULENT MIXING FOR FAR FROM EQUILIBRIUM PHENOMENA

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Abstract. The turbulence phenomenon is a modern component of fluid kinematics. In this area, the turbulent mixing is distinguished by its importance in applied engineering (including technical, social, economic applications). The turbulent mixing is an important feature of far from equilibrium models.Studying a mixing for a flow implies the analysis of successive stretching and folding phenomena for its particles, the influence of parameters and initial conditions, and also the issue of significant events – such as rare events - and their physical mean. A comparison between two and three dimensional flows produces useful remarks.

Key words:mixing, tendril-whorl flow, rotation, stretching, efficiency.

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1. The mixing concept

The turbulence term can be defined as "chaotic behavior of far from equilibrium systems, with very few freedom degrees". In this area there are two important theories:

a) The transition theory from smooth laminar flows to chaotic flows, characteristic to turbulence.

b) Statistic studies of the complete turbulent systems.

The statistical idea of flow is represented by the map:

$$x = \Phi_t(X), \; withX = \Phi_t(t=0)(X) \tag{1}$$

We say that X *is mapped in* x after a time t.

In the continuum mechanics the relation (1) is named *flow*, and it must be of class C^k. From the dynamic standpoint we have a map:

$$\Phi_t(X) \longrightarrow x \tag{2}$$

which is a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation:

$$0 \langle J \langle \infty, J = \det \left(\frac{\partial x_i}{\partial X_j} \right) \tag{3}$$

or, equivalently,

$$J = det(D\Phi_t(X)) \tag{4}$$

where D denotes the derivation with respect to the reference configuration, in this case X. The relation (3) implies two particles, X_1 and X_2, which occupy the same position x at a moment. Non-topological behavior (like break up, for example) *is not allowed*.

The basic measure for the deformation with respect to X is the *deformation gradient*, **F**:

$$\mathbf{F} = (X\Phi_t(\mathbf{X}))^T, F_{ij} = \left(\frac{\partial x_i}{\partial X_j} \right), \textit{ or } \mathbf{F} = D\Phi_t(\mathbf{X}) \tag{5}$$

where ∇_X denotes differentiation with respect to X. According to (3), **F** is non singular. The basic measure for the deformation with respect to x is the velocity gradient (∇_x denote differentiation with respect to x).

By differentiation of x with respect to X we obtain the relation:

$$dx_i = \frac{\partial x_i}{\partial X_j} \cdot dX_j \tag{6}$$

which gives the deformation of an infinitesimal filament of length $|dX|$ and orientation $M(= dX/|dX|)$ from its reference position, to the present position, dx , with the length $|dx|$ and the orientation $m(= dx/|dx|)$:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \tag{7}$$

This relation represents the basic deformation of a material filament. The corresponding relation for the area of an infinitesimal material surface is:

$$d\mathbf{a} = (\det \mathbf{F}) \cdot \left(\mathbf{F}^{-1}\right)^T \cdot d\mathbf{A}. \tag{8}$$

In this case in the present configuration we have the area $da = |da|$ and the orientation $n = da/|da|$, and in the reference configuration, the area $dA = |dA|$, and the orientation $N = dA/|dA|$.

Let us define the basic deformation measures: the *length deformation* λ and *surface deformation* η, with the relations [3,4]:

$$\lambda = \lim_{|d\mathbf{X}|\to 0}\frac{|d\mathbf{x}|}{|d\mathbf{X}|},\quad \eta = \lim_{|d\mathbf{A}|\to 0}\frac{|d\mathbf{a}|}{|d\mathbf{A}|}, \tag{9}$$

which are obtained from

$$\lambda = (C : MM)^{\frac{1}{2}},\ \eta = (\det F)\cdot\left(C^{-1} : NN\right)^{\frac{1}{2}}, \tag{10}$$

with $\mathbf{C}(= \mathbf{F}^T 2022\mathbf{F})$ the Cauchy-Green deformation tensor, and the vectors M, N defined by

$$\mathbf{M} = d\mathbf{X}/\left|d\mathbf{X}\right|,\ \mathbf{N} = d\mathbf{A}/\left|d\mathbf{A}\right| \tag{11}$$

The relation (10) has the scalar form:

$$\lambda = C_{ij}\cdot M_i\cdot N_j,\quad \eta = (\det F)\cdot\left(C_{ij}^{-1}\cdot N_i\cdot N_j\right),\ with\ \sum M_i^2 = 1,\ \sum N_j^2 = 1 \tag{12}$$

The deformation tensor $\mathbf{F}$ and the associated tensors $\mathbf{C}$, $\mathbf{C}^{-1}$ represent the basic quantities in the deformation analysis for the infinitesimal elements.

In this framework the mixing concept implies the stretching and folding of the material elements. If in an initial location P there is a material filament dX and an area element dA, the specific length and surface deformations are given by the relations:

$$\frac{D(\ln\lambda)}{Dt} = \mathbf{D} : \mathbf{mm},\quad \frac{D(\ln\eta)}{Dt} = \mathbf{v} - \mathbf{D} : \mathbf{nn} \tag{13}$$

where D is the deformation tensor, obtained by decomposing the velocity gradient in its symmetric and non-symmetric part:

$$\mathbf{v} = \mathbf{D} + \mathbf{\Omega}$$

$$\mathbf{D} = \frac{\left(\mathbf{v} + (\ \mathbf{v})^T\right)}{2} \tag{14}$$

the symmetric tensor

$$\mathbf{\Omega} = \frac{\left(\mathbf{v} - (\mathbf{v})^T\right)}{2}$$

the antisimetric tensor.

We say that the flow $\mathbf{x} = \Phi_t(\mathbf{X})$ has *a good mixing* if the mean values $D(ln\lambda)/Dt$ and $D(ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations $\mathbf{M}$ and $\mathbf{N}$.

As the above two quantities are bounded, the deformation efficiency can be naturally quantified. Thus, there is defined [3] the *deformation efficiency in length*, $e_\lambda = e_\lambda(X, M, t)$ of the material element dX, as

$$e_\lambda = \frac{D(ln\lambda)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1, \tag{15}$$

and similarly, the *deformation efficiency in surface*, $e_\eta = e_\eta(X, N, t)$ of the area element dA: in the case of an isochoric flow (the jacobian equal 1), we have:

$$e_\eta = \frac{D(ln\eta)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1. \tag{16}$$

2. The Tendril-Whorl flow. Analytical Features

2.1. The mathematical model

Two-dimensional flows increase their length by forming two basic kinds of structures: *tendrils* and *whorls* and their combinations. In complex two-dimensional fluid flows we can encounter tendrils within tendrils, whorls within whorls, and all other possible combinations. The tendril-whorl flow (TW) introduced by Khakhar, Rising and Ottino (1987) is a discontinuous succession of extensional flows and twist maps. In the simplest case all the flows are identical and the period of alternation extensional/rotational is also constant. But even the simplest case is complex enough and, on the other hand, it can be considered as the point of departure for several generalizations (smooth variation, distribution of time periods, etc). The physical motivation for this flow is that locally, a velocity field can be decomposed into extension and rotation. Also, from polar theorem point of view, a local deformation can be decomposed into stretching and rotation [3].

In the simplest case of the TW model, the velocity field over a single period is given by its extensional part:

$$v_x = -\varepsilon \cdot x,\ v_y = \varepsilon \cdot y,\ 0 < t < T_{ext} \tag{17}$$

and its rotational part:

$$v_r = 0,\ v_\theta = -\omega\left(r\right),\ T_{ext} < t < T_{ext} + T_{rot}, \tag{18}$$

where T_{ext} denotes the duration of the extensional component and T_{rot} the duration of rotational component.

The model consists of vortices producing whorls which are periodically squeezed by the hyperbolic flow leading to the formation of tendrils, and the process repeats. The function $\omega(r)$ is positive and specifies the rate of rotation. Its form is quite arbitrary and the most important aspect is that it has a maximum, that is, $\frac{d\omega(r)}{dr} = 0$ for some r. Independent of the form of $\omega(r)$, we can integrate the above velocity fields over one period and it gives:

$$f_{ext}\left(x, y\right) = \left(\frac{x}{\alpha}, \alpha \cdot y\right),\ f_{rot}\left(r, \theta\right) = \left(r, \theta + \triangle\theta\right) \tag{19}$$

where

$$\alpha = \exp\left(\varepsilon \cdot T_{ext}\right),\ \triangle\theta = -\tfrac{\omega(r) \cdot T_{rot}}{r}$$

corresponding to the extensional and rotational part.

2.2. Evaluating the efficiency of mixing.

Let us consider the extensional part (17) of TW model:

$$v_x = -\varepsilon \cdot x,\ v_y = \varepsilon \cdot y,\ 0 < t < T_{ext} \tag{20}$$

With the initial conditions $x\left(0\right) = X,\ y\left(0\right) = Y,$ the Cauchy problem has the solution :

$$x = X \cdot \exp\left(-\varepsilon \cdot T_{ext}\right),\ y = Y \cdot \exp\left(\varepsilon \cdot T_{ext}\right) \tag{21}$$

Following this solution, the deformation tensors $\mathbf{F}$,$\mathbf{C}$ and $\mathbf{C}^{-1}$ are quite easy to calculate. According to [3], the calculated expression (5) of deformation gradient is [2]:

$$\mathbf{F} = \begin{pmatrix} \exp\left(-\varepsilon \cdot T_{ext}\right) & 0 \\ 0 & \exp\left(\varepsilon \cdot T_{ext}\right) \end{pmatrix} \tag{22}$$

and the tensor $\mathbf{C}^{-1}$ is:

$$\mathbf{C}^{-1} = \begin{pmatrix} \exp\left(2\varepsilon \cdot T_{ext}\right) & 0 \\ 0 & \exp\left(-2\varepsilon \cdot T_{ext}\right) \end{pmatrix} \tag{23}$$

As it can be seen, the forms of these tensors are quite simple. In three dimensions, there were found rather complicated expressions [1], depending on few parameters.

In this context, the deformations λ^2 and η^2, in length and surface, have an appropriate form. It was found [2]:

$$\lambda^2 = \exp\left(-2\varepsilon \cdot T_{ext}\right) \cdot M_1^2 + \exp\left(2\varepsilon \cdot T_{ext}\right) \cdot M_2^2, \tag{24}$$

$$\eta^2 = \exp\left(2\varepsilon \cdot T_{ext}\right) \cdot N_1^2 + \exp\left(-2\varepsilon \cdot T_{ext}\right) \cdot N_2^2, \tag{25}$$

taking into account that $detF = 1$, where the versor condition

$$\sum M_i^2 = 1,\ \sum N_j^2 = 1$$

is satisfied.

The relation (24) shows that, for the extensional component of TW model, that is for a periodic flow, *the deformations in length and surface are less complex than for three-dimensional (non periodic) flow. Moreover, the expressions in length and surface are quite similar*. Therefore it is important to calculate and compare the deformation efficiencies e_λ and e_η.

A simple calculus makes easier [1] to use in practice the relation (15)-(16) for e_λ and e_η . We have:

$$e_\lambda = \frac{1}{2\lambda^2} \cdot \frac{d\lambda^2}{dt}, e_\eta = \frac{1}{2\eta^2} \cdot \frac{d\eta^2}{dt} \tag{26}$$

Applying these formulas, the deformation efficiencies are:

$$e_\lambda = 2\varepsilon \cdot \left(1 - \frac{2\exp(-2\varepsilon T_{ext}) \cdot M_1^2}{\exp(-2\varepsilon T_{ext}) \cdot M_1^2 + \exp(2\varepsilon T_{ext}) \cdot M_2^2}\right) \quad (27)$$

$$e_\eta = 2\varepsilon \cdot \left(1 - \frac{2\exp(-2\varepsilon T_{ext}) \cdot N_2^2}{\exp(-2\varepsilon T_{ext}) \cdot N_2^2 + \exp(2\varepsilon T_{ext}) \cdot N_1^2}\right) \quad (28)$$

where $M_1^2 + M_2^2 = 1$, $N_1^2 + N_2^2 = 1$.

The relations (27) and (28) give two functions of time, depending on the parameters ε, M_i, N_j, $0 < \varepsilon < 1$. T_{ext} represents the duration of the extensional component of TW model. It is a time period which can vary in a discrete range. Also, the relations show similar forms for the efficiencies, since λ^2 and η^2 are similar.

Let us consider $\varepsilon = 0.05$. For some discrete time moments $T_{ext} = 5, 10, 15, 20$ sec there were calculated the values for e_λ and e_η, for few versor values, taking into account the versor condition. For comparing the evolution of the functions e_λ and e_η, there were taken equal values for the length and surace vectors. The results of calculus are statistically presented in Table1.

Table 1.

Length/ surface vectors	T_{ext}=5sec.		T_{ext}=10sec		T_{ext}=15sec		T_{ext}=20sec	
	e_λ	e_η	e_λ	e_η	e_λ	e_η	e_λ	e_η
$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	0.046211	0.046211	0.076159	0.076159	0.090514	0.090514	0.096402	0.096402
$(1,0)$	-0.1	0.1	-0.1	0.1	-0.1	0.1	-0.1	0.1
$\left(\frac{1}{\sqrt{3}},\frac{\sqrt{2}}{\sqrt{3}}\right)$	0.068927	0.015223	0.073764	0.057397	0.095142	0.081888	0.098185	0.092932
$\left(\frac{1}{\sqrt{5}},\frac{\sqrt{2}}{\sqrt{5}}\right)$	0.083155	-0.019078	0.093454	0.029757	0.097541	0.066785	0.099088	0.086347
$\left(\frac{1}{\sqrt{7}},\frac{\sqrt{6}}{\sqrt{7}}\right)$	0.088445	-0.037641	0.095588	0.010374	0.098354	0.053997	0.099391	0.080210
$\left(\frac{1}{\sqrt{11}},\frac{\sqrt{10}}{\sqrt{11}}\right)$	0.092903	-0.057254	0.097329	-0.015014	0.099009	0.033523	0.099634	0.069039

3. Remarks

At this point a comparison with three-dimensional (non-periodic) flow is necessary. In [1] the analysis of the deformation efficiency in length and surface for a three-dimensional mathematical model associated to a vortex phenomena, for an aquatic algae (*Spirulina Platensis*), was realized. The experiments were realized with a special vortex tube [5]. For processing the algae Spirulina Platensis, there was used a non-dimensional parameter τ_a, given by:

$$\tau_a = \frac{t \cdot Q}{D^3},$$

where t represents the time (sec), D the diameter (m^3), and Q the debit (m^3/sec). By fragmenting the long chains of cellular filaments, there where obtained isolated cell units, or, as rare events, the break –up of the cell membrane (having a less than 100 angstrom).

The three-dimensional model considered:

$$\dot{x_1} = G \cdot x_2\ \dot{x_2} = K \cdot G \cdot x_1 \dot{x_3} = c,\ -1 < K < 1,\ c = \ const., \tag{29}$$

with the initial conditions

$$(27)\ x_1\,(0) = X_1,\ x_2\,(0) = X_2,\ x_3\,(0) = X_3,$$

a generalization to three dimensions of the two dimensional version used in [3], is a widespread model for isochoric flows.

The analysis of the structural stability for the problem (26)-(27) was in agreement with the experiments. It consists in studying the deformations in length and surface of the material filaments, with the vortex conditions imposed. The deformation tensor $\mathbf{F}$, and the tensors $\mathbf{C}, \mathbf{C}^{-1}$, had quite complicated expressions [1]. Also, the deformation eficiences in length and surface were quite complicated, depending on more parameters. The calculated expressions are the following:

$$e_\lambda = \frac{\dot{\gamma}}{2} \cdot \frac{\begin{array}{c}\left[\left(1+\frac{K}{4}\right) \cdot M_1^2 + \frac{1+K}{\sqrt{K}} \cdot M_1 M_2 + \left(1+\frac{1}{K}\right) \cdot M_2^2\right] \cdot \exp\left(2\dot{\gamma}t\right) + \\ \left[-\left(1+\frac{K}{4}\right) \cdot M_1^2 + \frac{1+K}{\sqrt{K}} \cdot M_1 M_2 - \left(1+\frac{1}{K}\right) \cdot M_2^2\right] \cdot \exp\left(-2\dot{\gamma}t\right)\end{array}}{\begin{array}{c}\left[\frac{1+K}{4} \cdot M_1^2 + \frac{1+K}{2\sqrt{K}} \cdot M_1 M_2 + \frac{1+K}{4K} \cdot M_2^2\right] \cdot \exp\left(2\dot{\gamma}t\right) + \\ \left[\frac{1+K}{4} \cdot M_1^2 - \frac{1+K}{2\sqrt{K}} \cdot M_1 M_2 + \frac{1+K}{4K} \cdot M_2^2\right] \cdot \exp\left(-2\dot{\gamma}t\right) + \\ \left(\frac{1-K}{2} \cdot M_1^2 - \frac{1+K}{\sqrt{K}} \cdot M_1 M_2 + \frac{K-1}{2K} \cdot M_2^2 + M_3^2\right)\end{array}} \tag{30}$$

$$e_\eta = \frac{\dot\gamma}{\det C}\cdot\frac{\left[\begin{array}{c} A(K)\cdot\left(2\left(6+\frac{1}{K}\cdot\right)\cdot\exp(2\dot\gamma t)-\frac{1}{2K}\cdot\exp(-2\dot\gamma t)+\frac{3K-2}{4K}\right)\cdot \\ (\exp(4\dot\gamma t)-\exp(-4\dot\gamma t))-\frac{17}{32}\cdot\left(2+\frac{1}{K}\right)\cdot\exp(4\dot\gamma t)+ \\ \frac{17}{64K}\cdot\exp(-4\dot\gamma t)\end{array}\right]\cdot \\ (N_1^2+N_2^2)+ \\ \left[\begin{array}{c}\left(2\cdot(6+\frac{1}{K}\right)\cdot\exp(2\dot\gamma t)-\frac{1}{2K}\cdot\exp(-2\dot\gamma t)+\frac{3K-2}{4K})\cdot \\ \exp(-4\dot\gamma t)\cdot\frac{(1+K)^2}{2K}-\left(\left(-2-\frac{1}{K}\right)\cdot\exp(2\dot\gamma t)+\frac{3}{2K}\cdot\right. \\ \exp(-2\dot\gamma t)-\frac{3K-2}{4K}\cdot\frac{(1+K)^2}{2K})\cdot\exp(-4\dot\gamma t)+\frac{(1+K)^2}{2K^2}- \\ \left(2+\frac{1}{K}\right)\cdot\frac{(1+K^2)}{2K}\end{array}\right]\cdot N_1N_2}{\begin{array}{c}[A(K)\cdot(\exp(4\dot\gamma t)+\exp(-4\dot\gamma t))+ \\ \frac{17}{64}\cdot(\exp(2\dot\gamma t)+\exp(-2\dot\gamma t))-B(K)]\cdot \\ (N_1^2+N_2^2)- \\ \left[\frac{1}{4}\cdot(\exp(4\dot\gamma t)+\exp(-4\dot\gamma t))-(\exp(2\dot\gamma t)+\exp(-2\dot\gamma t))+\frac{1}{2}\right]\cdot \\ N_1N_2\cdot\frac{(1+K)^2}{2K}+N_3^2\end{array}} \tag{31}$$

where $A(K)$ and $B(K)$ depend on K, and the parameters G and K were reunited in $\dot\gamma = G\cdot\sqrt{K}$.

Analysing this context, some remarks issue:

1) The amount of calculus for the extensional component of TW flow is less than in the case of three dimensional (non periodic) flow. Therefore, the behavior of the deformation efficiencies could expect to be smoother in the case of two-dimensional flow.

2) For the moment there were considered only four values for the period T_{ext}. If the function e_λ seems to be increasing, e_η has not a constant behavior. Therefore, it would be interesting to study the evolution for larger (discrete) values of T_{ext}.

3)An interesting fact is that, for equal values on both axes for the versors, e_λ and e_η *are equal*; therefore, a discrete plot for few period values would give new features, and new comparisons with results and experiments would appear;

4) A future aim is to construct an "extended" statistical table like Table1, taking into account more irrational versor values, and also more values for ε, for establishing more accurate properties for the functions e_λ and e_η . Also, searching for rare events, would give the possibility to compare the periodic two-dimensional flow with three-dimensional (no periodic) flow. In [1] there

were named *rare events* the events of breaking up the filaments of the aquatic algae, corresponding to the intrerruption of the simulation program.

5) A very important fact is that in [1] the mixing, and especially the turbulent mixing, occurs at *irrational values* $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ of the parameters and versors. This is not surprising, taking into account that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc can be considered themselves as *random values.* Therefore, the approaching of *random distributed events* could be favorable also for TW flow analysis.

6) The parameter T_{ext} can be measured in seconds, minutes or even in larger units, depending on the context. The same fact can be found also in [1] for three dimensional flow, where the turbulence occurs at *small values of the time units, being in agreement with experiments.* Therefore an analysis for larger T_{ext} would be useful.

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